GAUSSIAN FREE FIELDS AND KPZ RELATION IN $\mathbb{R}^4$

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Abstract. This work aims to extend part of the two dimensional results of Duplantier and Sheffield on Liouville quantum gravity [DS11] to four dimensions, and indicate possible extensions to other even-dimensional spaces $\mathbb{R}^{2n}$ as well as Riemannian manifolds.

Let Θ be the Gaussian free field on $\mathbb{R}^4$ with the underlying Hilbert space $H^2\left(\mathbb{R}^4\right)$ and the inner product $\left((I-\Delta)^2\cdot,\cdot\right)_{L^2}$, and θ a generic element from Θ. We consider a sequence of random Borel measures on $\mathbb{R}^4$, denoted by $\left\{m^\theta_{\epsilon_n}(dx): n\geq 1\right\}$, each of which is absolutely continuous with respect to the Lebesgue measure dx, and the density function is given by the exponential of a centered Gaussian family parametrized by $x\in\mathbb{R}^4$. We show that with probability 1, $m^\theta_{\epsilon_n}(dx)$ weakly converges as $\epsilon_n\downarrow 0$, and the limit measure can be "formally" written as "$m^\theta(dx)=e^{2\gamma\theta(x)}dx$". In this setting, we also prove a KPZ relation, which is the quadratic relation between the scaling exponent of a bounded Borel set on $\mathbb{R}^4$ under the Lebesgue measure and its counterpart under the random measure $m^\theta(dx)$.

Our approach is similar to the one used in [DS11] with adaptations to $\mathbb{R}^4$.

1. Introduction

Random measures have long been considered in 2-dimensional conformal field theory and quantum gravity since the work of Knizhnik, Polyakov and Zamolodchikov [KPZ]. Recently, a probabilistic proof of the formula due to Knizhnik, Polyakov and Zamolodchikov was given by Duplantier and Sheffield in [DS11]. On the unit planar disc $\mathbb{D}$, Duplantier and Sheffield construct the Liouville quantum gravity measure "$e^{\gamma h(z)}dz$", where dz is the Lebesgue measure on $\mathbb{D}$, γ is a properly chosen positive constant and h is an instance of the Gaussian free field (GFF) on $\mathbb{D}$ with the Dirichlet inner product. To be specific, they prove that the random measure exists as the weak convergence limit of $\epsilon^{\gamma^2/2}e^{\gamma h_\epsilon(z)}dz$ as $\epsilon\downarrow 0$, where $h_\epsilon(z)$ is the circular average of h over the circle centered at z with radius ϵ. They further show that there is a quadratic relation, known as the KPZ relation, between the scaling exponent of a random set under the Lebesgue measure, and its counterpart under the quantum gravity measure. Another derivation was given in [DB]. Gaussian free field in dimension 2 has also been considered in [HMP] and numerous other papers.

In this article, we generalize part of the results from [DS11] to four dimensions. We define the Euclidean GFF on $\mathbb{R}^4$, denoted by Θ, with the inner product determined by the Bessel operator $(I-\Delta)^2$. In other words, the underlying Hilbert space

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of Θ is given by the Sobolev space $H^2\left(\mathbb{R}^4\right)$ with the inner product $\left((I-\Delta)^2\cdot,\cdot\right)_{L^2}$. In this setting, we prove (Section 2, Theorem 5) that given $0<\gamma^2<2\pi^2$, almost every $\theta\in\Theta$ admits a random measure on $\mathbb{R}^4$ which "formally" has the density $e^{2\gamma\theta(x)}$ with respect to the Lebesgue measure dx on $\mathbb{R}^4$. We also show that this random measure satisfies a KPZ relation similar to the one in the two dimensional case. Namely, if $\kappa\in[0,1]$ is the scaling exponent of a bounded Borel set in $\mathbb{R}^4$ under the Lebesgue measure, and $K\in[0,1]$ is the scaling exponent of the same set but under the random measure obtained above (both κ and K will be defined in Section 4), then κ and K satisfy the following quadratic relation (Section 4, Theorem 9):

$$\kappa=K\left(1-\frac{\gamma^2}{16\pi^2}\right)+\frac{\gamma^2}{16\pi^2}K^2.$$

Our proof follows the outline of the proof in [DS11] with adaptations to four dimensions. Mainly we have to overcome (both in "designing" the model to work with and in technical details) the difficulties caused by the absence in our problem the two dimensional conformal structure. To interpret rigorously an instance θ of the GFF on the entire Euclidean space $\mathbb{R}^4$, we adopt the theory of the abstract Wiener space. A key ingredient in this theory is the underlying Hilbert space whose inner product determines the covariance structure of the field. It is already known that in order to obtain a measure which "formally" has the exponential of $\theta(x)$ as the density with respect to dx, the covariance function $\operatorname{Cov}(\theta(x),\theta(y))$ can at most grow at the rate of $-\log|x-y|$ when $|x-y|$ is small. Taking this into account, $H^2\left(\mathbb{R}^4\right)$ with the inner product $\left((I-\Delta)^2\cdot,\cdot\right)_{L^2}$ becomes our natural choice. Also this way of defining the GFF makes it possible, in certain situations, to obtain explicit formulas of the covariance function. To construct the random measure and thereafter to study it, we always need to relate it to a sequence of approximating measures which converges in some proper sense. So it is our intention to choose the approximating measures appropriately so they will be convenient to work with. In the two dimensional case in [DS11], the approximating measures are in terms of the circular averages of the GFF on $\mathbb{D}$. In fact, the properties of the Gaussian family consisting of these circular averages play an important role in the proof. For example, if h is an instance of the GFF on $\mathbb{D}$, then given any $z\in\mathbb{D}$, the one-parameter family $\{h_\epsilon(z):0<\epsilon\le 1\}$ has up to a time change the same distribution as a standard Brownian motion. Such properties are derived from the Green's function of the Laplace operator Δ on $\mathbb{D}$, which, in particular, is harmonic. Therefore, it should not be surprising that the trivial analogue in four dimensions, that is, the family of spherical averages of θ, fails to have such properties, which makes it a less than optimal substitute for h_ϵ in carrying out this project on $\mathbb{R}^4$. In Section 2, we present one possible replacement for h_ϵ in four dimensions which still has simple and concrete geometric interpretations (in fact, it is given by a functional of the spherical average of θ), but possesses, to a large extent, similar properties to those of h_ϵ in two dimensions. In Section 3, we use the results from Section 2 to build the approximating measures, and then prove they almost surely admit a limit measure in the sense of weak convergence. In Section 4, we lay out an outline to derive the KPZ relation and the proofs of the main results are collected in Section 5. Other work on the KPZ relation in higher dimensions with different settings can be found in [JJRV, RV].

Our original interest in constructing such a random measure lies in its potential applications in the study of conformal classes of Riemannian metrics. In fact, another more geometric point of view on the GFF on a planar domain or more generally on a surface Σ, is to consider it as a measure on a conformal class of metrics on Σ, where the measure is constructed with the help of a reference metric g_0 on Σ, but where the result does not depend on g_0. It seems natural to generalize this approach to conformal classes of metrics on higher-dimensional manifolds. It turns out that on a compact four-dimensional manifold $\mathfrak{M}$, a natural replacement for the Laplace-Beltrami operator Δ (that is used in the construction of the GFF on surfaces) is the 4-th order *Paneitz operator* that arises in the conformal geometry. More generally, on compact $2n$-dimensional manifolds it seems natural to use the *dimension-critical GJMS operator* in the construction of higher-dimensional analogues of the two-dimensional GFF. This will be further explained in the second part of Section 6.

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2. Spherical Averages of GFF on $\mathbb{R}^4$

We start with a brief review of some fundamental facts about the abstract Wiener space theory ([Gr] or [S08]). An abstract Wiener space is commonly used in constructions of infinite dimensional Gaussian measures. The basic setting of an abstract Wiener space is as follows. Given a (infinite dimensional) Banach space Θ and a (infinite dimensional) Hilbert space H, assume both Θ and H are separable, and H can be continuously embedded into Θ as a dense subspace. Therefore if x^* is a bounded linear functional on Θ (denoted by $x^* \in \Theta^*$), then there is unique $h_{x^*} \in H$ such that for every $h \in H$, $(h, h_{x^*})_H = \langle h, x^* \rangle$, where $\langle \cdot, * \rangle$ refers to the action of Θ^* on Θ (or more specifically in later discussions, the action of tempered distributions on test functions). Let $\mathcal{W}$ be a probability measure on $(\Theta, \mathfrak{B}_\Theta)$ where $\mathfrak{B}_\Theta$ is the Borel $\sigma-$algebra of Θ. If $\mathcal{W}$ satisfies

$$\mathbb{E}^{\mathcal{W}}\left[\exp\left(i\left\langle \cdot, x^*\right\rangle\right)\right] = \exp\left(-\frac{\|h_{x^*}\|_H^2}{2}\right) \text{ for all } x^* \in \Theta^*,$$

then the triple $(H, \Theta, \mathcal{W})$ is called an *abstract Wiener space*. It is known ([S11], §8.3) that given any separable Hilbert space, one can always find Θ and $\mathcal{W}$ such that $(H, \Theta, \mathcal{W})$ forms an abstract Wiener space. Moreover, since $\{h_{x^*} : x^* \in \Theta^*\}$ is also dense in H, the linear mapping

$$\mathcal{I}: \ h_{x^*} \in H \mapsto \mathcal{I}\left(h_{x^*}\right) \equiv \left\langle \cdot, x^*\right\rangle \in L^2\left(\mathcal{W}\right)$$

can be uniquely extended as a linear isometry from H to $L^2(\mathcal{W})$. Its images $\{\mathcal{I}(h) : h \in H\}$, known as the *Paley-Wiener integrals*, form a centered Gaussian family whose covariance is given by

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_1\right)\mathcal{I}\left(h_2\right)\right] = \left(h_1, h_2\right)_H \text{ for all } h_1, h_2 \in H.$$

We point out that although the Hilbert structure of H plays an essential role, $\mathcal{W}(H) = 0$ and the choice of Θ is not unique.

As we have mentioned in the previous section, we consider in our project the infinite dimensional Gaussian measure on the space of certain tempered distributions on $\mathbb{R}^4$, with the underlying Hilbert space given by the Sobolev space $H \equiv H^2\left(\mathbb{R}^4\right)$, which is the completion of the real valued Schwartz test function space $\mathcal{S}\left(\mathbb{R}^4\right)$ under the inner product

$$(f_1, f_2)_H \equiv \int_{\mathbb{R}^4} (I-\Delta)^2 f_1(x) f_2(x)\, dx \text{ for all } f_1, f_2 \in \mathcal{S}\left(\mathbb{R}^4\right).$$

Then, given this particular choice of H, our notion of the *Gaussian free field* on $\mathbb{R}^4$ refers to any probability space $(\Theta, \mathfrak{B}_\Theta, \mathcal{W})$ such that $(\Theta, H, \mathcal{W})$ forms an abstract Wiener space. For example, if $\tilde{\Theta}$ is the space of continuous functions $\theta: \mathbb{R}^4 \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} (\log(e+|x|))^{-1} |\theta(x)| = 0,$$

then Θ can be chosen as the image of $\tilde{\Theta}$ under the Bessel operator $(I-\Delta)^{\frac{1}{4}}$, i.e.,

$$\Theta = \left\{ (I-\Delta)^{\frac{1}{4}} \theta : \theta \in \tilde{\Theta} \right\}.$$

From this we observe that Θ consists of tempered distributions which in general are not defined point-wise. Nonetheless, we can understand some properties of the GFF by studying the Paley-Wiener integrals, which can be viewed as "generalized" action of certain tempered distributions on Θ.

In addition, if $H^{-2} = H^{-2}\left(\mathbb{R}^4\right)$ is the Hilbert space consisting of tempered distributions μ such that

$$\|\mu\|_{H^{-2}}^2 \equiv \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \left(1+|\xi|^2\right)^{-2} |\hat{\mu}(\xi)|^2\, d\xi < \infty$$

where $\hat{\mu}$ is the Fourier transform (without the factor $(2\pi)^{-2}$ in the definition) of μ, then we can identify H with H^{-2} since $(I-\Delta)^{-2}: H^{-2} \to H$ is obviously a linear isometry. We will abuse the notation[1] by denoting "h_ν" the image of $\nu \in H^{-2}$ under $(I-\Delta)^{-2}$. Then h_ν is the unique element in H such that $\langle h, \nu \rangle = (h, h_\nu)_H$ for all $h \in H$, which suggests that the corresponding Paley-Wiener integral $\mathcal{I}(h_\nu)$ can be viewed as a "representation" of the action of ν on Θ, even though ν is not in Θ^* and $\mathcal{I}(h_\nu)(\theta)$ is only defined for almost every $\theta \in \Theta$. Meanwhile, $\left\{\mathcal{I}(h_\nu) : \nu \in H^{-2}\right\}$ is also a Gaussian family whose covariance is given by

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}(h_{\nu_1}) \mathcal{I}(h_{\nu_2})\right] = (h_{\nu_1}, h_{\nu_2})_H = (\nu_1, \nu_2)_{H^{-2}}.$$

With these in mind, as a natural analogue of the 2D circular average, we consider the spherical average of the GFF on $\mathbb{R}^4$. To this end, for every $x \in \mathbb{R}^4$ and $\epsilon > 0$, denote σ_ϵ^x the tempered distribution determined by

$$\langle f, \sigma_\epsilon^x \rangle \equiv \frac{1}{2\pi^2 \epsilon^3} \int_{S_\epsilon(x)} f(y)\, d\sigma(y) \text{ for all } f \in \mathcal{S}\left(\mathbb{R}^4\right),$$

where $S_\epsilon(x)$ is the sphere centered at x with radius ϵ, and $d\sigma$ is the surface area measure on $S_\epsilon(x)$. Clearly, the action of σ_ϵ^x is to take the spherical average of f

[1] The subscript of "h_ν" is an element of H^{-2}, not to be confused with "h_{x^*}" in the definition of the abstract Wiener space where $x^* \in \Theta^*$.

over $S_\epsilon(x)$. It is an easy matter to verify that $\sigma_\epsilon^x \in H^{-2}$. In fact, one only needs to write down the Fourier transform of σ_ϵ^x as

$$\hat{\sigma_\epsilon^x}(\xi) = 2(\epsilon|\xi|)^{-1} J_1(\epsilon|\xi|) e^{i(x,\xi)_{\mathbb{R}^4}} \tag{2.1}$$

where $J_k(r)$ is the Bessel function of order $k \in \mathbb{N}$, and use the fact that $J_k(r)$ is asymptotic to $r^{-1/2}$ when r is large.

As we have indicated in the introduction (and as we will confirm in the next lemma), the spherical average of the GFF on $\mathbb{R}^4$ does not behave as "nicely" as the circular average of the GFF in two dimensions. For one thing $\left\{\mathcal{I}\left(h_{\sigma_\epsilon^x}\right) : \epsilon > 0\right\}$ fails to be a reversed Markov process. An intuitive way to view this is that, the spherical average does not bear enough information in itself for this Gaussian family parametrized by radius $\epsilon > 0$ to be (reversed) Markovian. It might also be helpful to relate this to the following analogous problem: when solving PDEs with higher order differential operator on a domain with boundary, one often needs more than one boundary condition (e.g., both the Dirichlet and the Neumann boundary conditions) to uniquely determine the solution. Inspired by this idea, besides the average itself we will also "collect" one more piece of information about the GFF from each sphere, which is the "derivative" of the average with respect to the radius. Namely, for every $x \in \mathbb{R}^4$ and $\epsilon > 0$, denote $d\sigma_\epsilon^x$ the tempered distribution given by $\langle f, d\sigma_\epsilon^x\rangle \equiv \frac{d}{d\epsilon}\langle f, \sigma_\epsilon^x\rangle$ for all $f \in \mathcal{S}\left(\mathbb{R}^4\right)$, then the action of $d\sigma_\epsilon^x$ can be viewed as to take the derivative of the spherical average of the GFF in the radial direction. It follows trivially from (2.1) that

$$\hat{d\sigma_\epsilon^x}(\xi) = \frac{d}{d\epsilon}\hat{\sigma_\epsilon^x}(\xi) = -2\epsilon^{-1} J_2(\epsilon|\xi|) e^{i(x,\xi)_{\mathbb{R}^4}}. \tag{2.2}$$

In particular, $d\sigma_\epsilon^x$ is also in H^{-2} and so $\left\{\mathcal{I}\left(h_{\sigma_\epsilon^x}\right), \mathcal{I}\left(h_{d\sigma_\epsilon^x}\right) : x \in \mathbb{R}^4, \epsilon > 0\right\}$ forms a centered Gaussian family whose covariance is determined by the H^{-2} inner product of $\left\{\sigma_\epsilon^x, d\sigma_\epsilon^x : x \in \mathbb{R}^4, \epsilon > 0\right\}$.

The next lemma in some sense validates our decision to take $d\sigma_\epsilon^x$ into account. It shows that by putting $\mathcal{I}\left(h_{\sigma_\epsilon^x}\right)$ and its "derivative" $\mathcal{I}\left(h_{d\sigma_\epsilon^x}\right)$ together[2], not only does the Gaussian family recover the reversed Markov property in the concentric case (with $x \in \mathbb{R}^4$ fixed, parametrized by $\epsilon > 0$ only), the non-concentric family (parametrized by both $\epsilon > 0$ and $x \in \mathbb{R}^4$) also resembles, to a large extent, its counterpart in two dimensions. To be precise, we define the vector-valued Gaussian random variable:

$$V_\epsilon^x \equiv \begin{pmatrix} \mathcal{I}\left(h_{\sigma_\epsilon^x}\right) \\ \mathcal{I}\left(h_{d\sigma_\epsilon^x}\right) \end{pmatrix} \text{ for every } x \in \mathbb{R}^4 \text{ and } \epsilon > 0.$$

Then, under certain circumstances, the covariance matrix of the Gaussian family $\left\{V_\epsilon^x : x \in \mathbb{R}^4, \epsilon > 0\right\}$ can be evaluated explicitly as follows.

Lemma 1. *For $r \in (0, \infty)$, define the following four matrices:*

$$\mathbf{A}(r) \equiv \begin{pmatrix} K_1'(r) & K_1(r)/r \\ K_1''(r) & -K_2(r)/r \end{pmatrix}, \mathbf{B}(r) \equiv \begin{pmatrix} I_1(r)/r & I_1'(r) \\ I_2(r)/r & I_1''(r) \end{pmatrix},$$

$$\mathbf{C}(r) \equiv \begin{pmatrix} I_1(r)/r & 0 \\ I_2(r) & I_1(r)/r \end{pmatrix}, \ \mathbf{D}(r) \equiv \begin{pmatrix} -K_2(r) & K_1(r)/r \\ K_1(r)/r & 0 \end{pmatrix},$$

[2]This idea came from discussions with Daniel W. Stroock when the first author was studying at MIT.

where I_k, K_k are the modified Bessel functions of order $k \in \mathbb{N}$. Then,

(1), given $x \in \mathbb{R}^4$ and $\epsilon_1 \geq \epsilon_2 > 0$,

$$\mathbb{E}^{\mathcal{W}}\left[V_{\epsilon_1}^x \left(V_{\epsilon_2}^x\right)^\top\right] = \left(-\frac{1}{4\pi^2}\right)\mathbf{A}\left(\epsilon_1\right)\mathbf{B}^\top\left(\epsilon_2\right). \tag{2.3}$$

In particular, $\{V_\epsilon^x : \epsilon > 0\}$ is a vector-valued Gaussian reversed Markov process in the sense that for every Borel $A \subseteq \mathbb{R}^2$,

$$\mathcal{W}\left(V_{\epsilon_2}^x \in A | \sigma\left\{V_\eta^x : \eta \geq \epsilon_1\right\}\right) = \mathcal{W}\left(V_{\epsilon_2}^x \in A | V_{\epsilon_1}^x\right),$$

where $\sigma\left\{V_\eta^x : \eta \geq \epsilon_1\right\}$ is the $\sigma-$algebra generated by $\left\{V_\eta^x : \eta \geq \epsilon_1\right\}$.

(2), given $x, y \in \mathbb{R}^4$, $x \neq y$, and $\epsilon_1, \epsilon_2 > 0$ with $\epsilon_1 > |x-y| + \epsilon_2$,

$$\mathbb{E}^{\mathcal{W}}\left[V_{\epsilon_1}^x \left(V_{\epsilon_2}^y\right)^\top\right] = \left(-\frac{1}{2\pi^2}\right)\mathbf{A}\left(\epsilon_1\right)\mathbf{C}\left(|x-y|\right)\mathbf{B}^\top\left(\epsilon_2\right). \tag{2.4}$$

(3), given $x, y \in \mathbb{R}^4$, $x \neq y$, and $\epsilon_1, \epsilon_2 > 0$ with $|x-y| > \epsilon_1 + \epsilon_2$,

$$\mathbb{E}^{\mathcal{W}}\left[V_{\epsilon_1}^x \left(V_{\epsilon_2}^y\right)^\top\right] = \left(-\frac{1}{2\pi^2}\right)\mathbf{B}\left(\epsilon_1\right)\mathbf{D}\left(|x-y|\right)\mathbf{B}^\top\left(\epsilon_2\right). \tag{2.5}$$

The proof of (2.3)-(2.5) relies heavily on the integral formulas and identities of Bessel functions. The complete detailed computations are given in the appendix. Here we only make the following observations.

First, we claim that the distribution of $\left\{V_\epsilon^x : x \in \mathbb{R}^4, \epsilon > 0\right\}$ is invariant under isometries in spatial variables in the sense that $\left\{V_\epsilon^{\mathcal{T}(x)} : x \in \mathbb{R}^4, \epsilon > 0\right\}$ has exactly the same distribution as $\left\{V_\epsilon^x : x \in \mathbb{R}^4, \epsilon > 0\right\}$ for every $\mathcal{T} : \mathbb{R}^4 \to \mathbb{R}^4$ satisfying $|\mathcal{T}(x) - \mathcal{T}(y)| = |x-y|$ for all $x, y \in \mathbb{R}^4$. Perhaps the most straightforward way to see this is to write down the covariance matrix of the family, or equivalently, the H^{-2} inner product of $\left\{\sigma_\epsilon^x, d\sigma_\epsilon^x : x \in \mathbb{R}^4, \epsilon > 0\right\}$ in the integral form using (2.1) and (2.2). We will actually do this in the appendix (formulas (7.1)-(7.6)). The result shows that the only dependence of the covariance matrix on spatial variables is through the distance between centers of the spheres that are involved.

Second, we point out that all the matrices above $\mathbf{A}(r)$, $\mathbf{B}(r)$, $\mathbf{C}(r)$ and $\mathbf{D}(r)$ are invertible for all $r > 0$. This fact can certainly be verified by direct computations using the explicit formulas given above, but it also follows, more generally, from the simple fact that $d\sigma_\epsilon^x$ is linearly independent of σ_ϵ^y for every $x, y \in \mathbb{R}^4$ and $\epsilon > 0$. Therefore, assuming (2.3) is true, then given x fixed and $\epsilon_1 \geq \epsilon_2 > 0$, the conditional expectation of $V_{\epsilon_2}^x$ conditioning on $V_{\epsilon_1}^x$ equals

$$\mathbb{E}^{\mathcal{W}}\left[V_{\epsilon_2}^x \left(V_{\epsilon_1}^x\right)^\top\right]\left(\mathbb{E}^{\mathcal{W}}\left[V_{\epsilon_1}^x \left(V_{\epsilon_1}^x\right)^\top\right]\right)^{-1} V_{\epsilon_1}^x = \mathbf{B}\left(\epsilon_2\right)\mathbf{B}^{-1}\left(\epsilon_1\right)V_{\epsilon_1}^x.$$

On the other hand, we observe that for all $\eta \geq \epsilon_1$,

$$\mathbb{E}^{\mathcal{W}}\left[\left(V_{\epsilon_2}^x - \mathbf{B}\left(\epsilon_2\right)\mathbf{B}^{-1}\left(\epsilon_1\right)V_{\epsilon_1}^x\right)\left(V_\eta^x\right)^\top\right] = 0.$$

This means,

$$V_{\epsilon_2}^x - \mathbf{B}\left(\epsilon_2\right)\mathbf{B}^{-1}\left(\epsilon_1\right)V_{\epsilon_1}^x$$

is independent of V_η^x for all $\eta \geq \epsilon_1$, which certainly implies the reversed Markov property.

Next, we observe that under the circumstances as prescribed in Lemma 1, the covariance matrix of $\left\{V_\epsilon^x : x \in \mathbb{R}^4, \epsilon > 0\right\}$ is "separable" in the sense that it splits into factors each of which only depends on one of the variables ϵ_1, ϵ_2 and $|x-y|$. A second look at the formulas (2.3)-(2.5) suggests that we should "normalize" V_ϵ^x by $\mathbf{B}^{-1}(\epsilon)$. Namely, if denote $U_\epsilon^x \equiv \mathbf{B}^{-1}(\epsilon) V_\epsilon^x$, then the previous observations imply that given x fixed, $\{U_\epsilon^x : \epsilon > 0\}$ is a vector-valued Gaussian process with independent (reversed) increment whose distribution does not depend on x. Moreover, (2.4) and (2.5) show that $\mathbb{E}^{\mathcal{W}}\left[U_{\epsilon_1}^x \left(U_{\epsilon_2}^y\right)^\top\right]$ only depends on ϵ_1 and $|x-y|$ when[3] $\overline{B_{\epsilon_2}(y)} \subsetneq B_{\epsilon_1}(x)$, and the same matrix only depends on $|x-y|$ when $\overline{B_{\epsilon_2}(y)} \cap \overline{B_{\epsilon_1}(x)} = \emptyset$. Because U_ϵ^x has these properties, we are one step closer to finding a plausible replacement for the circular average of the two dimensional GFF.

Clearly, for any constant $\zeta = (\zeta_1, \zeta_2)^\top \in \mathbb{R}^2$, $(U_\epsilon^x, \zeta)_{\mathbb{R}^2}$ is a scalar valued Gaussian random variable (in fact, it is a Paley-Wiener integral), which, when parametrized by $x \in \mathbb{R}^4$ and $\epsilon > 0$, forms a Gaussian family that preserves the properties described above. Our goal is to find a proper $\zeta \in \mathbb{R}^2$ such that the random variable

$$\theta \in \Theta \mapsto (U_\epsilon^x, \zeta)_{\mathbb{R}^2}(\theta) = \zeta^\top \mathbf{B}^{-1}(\epsilon) V_\epsilon^x(\theta)$$

becomes a "legitimate" approximation for a multiple of the value of θ at point x for every $x \in \mathbb{R}^4$. Namely, we want to choose ζ so that if $\mu_\epsilon^x \in H^{-2}$ is given by

$$\mu_\epsilon^x \equiv \zeta^\top \mathbf{B}^{-1}(\epsilon) \begin{pmatrix} \sigma_\epsilon^x \\ d\sigma_\epsilon^x \end{pmatrix}, \tag{2.6}$$

then μ_ϵ^x converges to a constant multiple of the point mass δ_x at x as $\epsilon \downarrow 0$ in the sense of tempered distribution. We can reach this goal by writing down the formula of $\mathbf{B}^{-1}(\epsilon)$ explicitly and examining the asymptotics of the Bessel functions near the origin (detailed computations are given in the appendix). As a result, we find that $\zeta = (1,1)^\top$ will serve the purpose, in which case $\mu_\epsilon^x \to 2\delta_x$ as $\epsilon \downarrow 0$ for every $x \in \mathbb{R}^4$. From now on, we will assume μ_ϵ^x is as in (2.6) with $\zeta = (1,1)^\top$. Since $\mathcal{I}\left(h_{\mu_\epsilon^x}\right) = \zeta^\top \mathbf{B}^{-1}(\epsilon) V_\epsilon^x$, we can transfer the results in Lemma 1 to the Gaussian family $\left\{\mathcal{I}\left(h_{\mu_\epsilon^x}\right) : x \in \mathbb{R}^4, \epsilon > 0\right\}$.

Theorem 2. *Define the positive function $G : r \in (0,\infty) \mapsto G(r) \in (0,\infty)$ by*

$$G(r) \equiv \left(-\frac{1}{4\pi^2}\right) \frac{2I_1(r)K_1(r) + 2I_2(r)K_0(r) - 1}{I_1^2(r) - I_0(r)I_2(r)}. \tag{2.7}$$

Then, we have
(1), given $x \in \mathbb{R}^4$ and $\epsilon_1 \geq \epsilon_2 > 0$,

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\mu_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{\mu_{\epsilon_2}^x}\right)\right] = \mathbb{E}^{\mathcal{W}}\left[\mathcal{I}^2\left(h_{\mu_{\epsilon_1}^x}\right)\right] = G(\epsilon_1). \tag{2.8}$$

In particular, $\left\{\mathcal{I}\left(h_{\mu_\epsilon^x}\right) : \epsilon > 0\right\}$ is a Gaussian process with independent reversed increments in the sense that $\mathcal{I}\left(h_{\mu_{\epsilon_2}^x}\right) - \mathcal{I}\left(h_{\mu_{\epsilon_1}^x}\right)$ is independent of $\sigma\left\{\mathcal{I}\left(h_{\mu_\eta^x}\right) : \eta \geq \epsilon_1\right\}$.
(2), given $x, y \in \mathbb{R}^4$, $x \neq y$, and $\epsilon_1, \epsilon_2 > 0$ with $\epsilon_1 > |x-y| + \epsilon_2$,

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\mu_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{\mu_{\epsilon_2}^y}\right)\right] = I_0(|x-y|)G(\epsilon_1) - \frac{1}{4\pi^2}\frac{I_2(|x-y|)}{I_1^2(\epsilon_1) - I_0(\epsilon_1)I_2(\epsilon_1)}. \tag{2.9}$$

[3] The notation "$B_r(x)$" ("$\overline{B_r(x)}$") denotes the open (closed) ball centered at x with radius r.

(3), given $x, y \in \mathbb{R}^4$, $x \neq y$, and $\epsilon_1, \epsilon_2 > 0$ with $|x-y| > \epsilon_1 + \epsilon_2$,

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\mu^y_{\epsilon_2}}\right)\right] = \frac{1}{2\pi^2} K_0\left(|x-y|\right). \tag{2.10}$$

There is not much to be said about the proof since everything follows from straightforward computations and the results in Lemma 1. However, we should point out the following facts.

First, it is an easy matter to check that $G : (0, \infty) \to (0, \infty)$ is smooth and strictly decreasing with $\lim_{r\downarrow 0} G(r) = +\infty$ and $\lim_{r\uparrow\infty} G(r) = 0$. Therefore, G^{-1} is defined and also strictly decreasing on $(0, \infty)$. Fix a positive constant R, and for every $r \in (0, R]$, define

$$0 \le t \equiv G(r) - G(R) \text{ and } X_t \equiv \mathcal{I}\left(h_{\mu^x_{G^{-1}(t+G(R))}}\right) - \mathcal{I}\left(h_{\mu^x_R}\right). \tag{2.11}$$

Then $\{X_t : t \ge 0\}$, as a Gaussian process on Θ under $\mathcal{W}$, has the same distribution as the standard Brownian motion, which in particular is independent of the choice of x. In other words, $\left\{\mathcal{I}\left(h_{\mu^x_\epsilon}\right) : 0 < \epsilon \le R\right\}$ has the distribution of a Brownian motion up to a non-random time change.

Second, the formulas (2.9) and (2.10) indicate that the covariance function does not depend on ϵ_2 when $\overline{B_{\epsilon_2}(y)} \subseteq B_{\epsilon_1}(x)$ and does not even depend on ϵ_1 or ϵ_2 when $\overline{B_{\epsilon_2}(y)} \cap \overline{B_{\epsilon_1}(x)} = \emptyset$. If one does the calculation with circular averages of the two dimensional GFF in each case analogously, one would see the same phenomenon. Namely, the smaller radius does not appear in the covariance function if one circle is entirely contained in the disk bounded by the other circle, while neither of the radii matters if the two disks bounded by the two circles respectively don't intersect. Such properties are consequences of the mean value theorem applied to the Green's function of the Laplace operator Δ in two dimensions, and these special properties of harmonic functions are no longer available to us in four dimensions. Nonetheless, we have seen from the above that by substituting $\left\{\mathcal{I}\left(h_{\mu^x_\epsilon}\right) : x \in \mathbb{R}^4, \epsilon > 0\right\}$ for the circular average, we will recover in the four-dimensional setting properties similar to those in two dimensions.

Finally, by examining the asymptotics of the Bessel functions near the origin, one finds that function G as defined in (2.7) is asymptotic to $-\frac{1}{2\pi^2}\log r$ when r is small. Therefore, in both case (1) and case (2) from above, the covariance function is asymptotic to $-\frac{1}{2\pi^2}\log\epsilon_1$ for sufficiently small ϵ_1, while in case (3), the right hand side of (2.10) is asymptotic to $-\frac{1}{2\pi^2}\log|x-y|$ for sufficiently small $|x-y|$. In this sense, the covariance function of $\left\{\mathcal{I}\left(h_{\mu^x_\epsilon}\right) : x \in \mathbb{R}^4, \epsilon > 0\right\}$ does have logarithmic growth near diagonal as one would have expected.

By now, one should be able to believe that $\left\{\mathcal{I}\left(h_{\mu^x_\epsilon}\right) : x \in \mathbb{R}^4, \epsilon > 0\right\}$ is a reasonable replacement for the circular average of the 2D GFF in order to construct a random measure on $\mathbb{R}^4$. Indeed, the construction based on this Gaussian family will be carried out in the next section. We close this section with an important observation about $\mathcal{I}\left(h_{\mu^x_\epsilon}\right)$ as a mapping from the variable $x \in \mathbb{R}^4$ to a random variable on Θ under $\mathcal{W}$.

Corollary 3. *Given $\epsilon > 0$, the mapping $x \in \mathbb{R}^4 \mapsto \mathcal{I}\left(h_{\mu^x_\epsilon}\right) \in L^2(\mathcal{W})$ is continuous. Moreover, if $\alpha \in \left(0, \frac{1}{2}\right)$, then for almost every $\theta \in \Theta$, $x \in \mathbb{R}^4 \mapsto \mathcal{I}\left(h_{\mu^x_\epsilon}\right)(\theta) \in \mathbb{R}$ is Hölder continuous with Hölder constant α.*

Proof. By Kolmogorov's continuity criterion ([S11], §4.3) applied to Gaussian random variables, in order to prove both statements in Corollary 3, it would be sufficient if we can show that there exist constant $\beta > 0$ and $0 < C_{\beta,\epsilon} < \infty$ such that for every $x, y \in \mathbb{R}^4$,

$$\left\|\mathcal{I}\left(h_{\mu_\epsilon^x}\right) - \mathcal{I}\left(h_{\mu_\epsilon^y}\right)\right\|_{L^2(\mathcal{W})}^2 \leq C_{\beta,\epsilon}\,|x-y|^\beta .$$

To simplify the notations, we write μ_ϵ^x as $\mu_\epsilon^x \equiv f_1(\epsilon)\,\sigma_\epsilon^x + f_2(\epsilon)\,d\sigma_\epsilon^x$, where both f_1 and f_2 are actually analytic functions in $\epsilon \in [0, R]$ (the explicit formulas for f_1 and f_2 are given by (7.9) in the appendix). Therefore, we only need to show that both $\|\sigma_\epsilon^x - \sigma_\epsilon^y\|_{H^{-2}}^2$ and $\|d\sigma_\epsilon^x - d\sigma_\epsilon^y\|_{H^{-2}}^2$ are bounded by $C_{\beta,\epsilon}\,|x-y|^\beta$. Perhaps the most straightforward way to see this is writing down the integral expressions for $\|\sigma_\epsilon^x - \sigma_\epsilon^y\|_{H^{-2}}^2$ and $\|d\sigma_\epsilon^x - d\sigma_\epsilon^y\|_{H^{-2}}^2$. Again, we refer to the formulas (7.1)-(7.6) in the appendix. From there, together with the series expression for the Bessel functions ([Wat], §3.1), it is an easy matter to check that one can get the desired upper bound for $\|\sigma_\epsilon^x - \sigma_\epsilon^y\|_{H^{-2}}^2$ and $\|d\sigma_\epsilon^x - d\sigma_\epsilon^y\|_{H^{-2}}^2$ so long as $\beta \in (0,1)$. □

3. Construction of Random Measure

In this section, we will use the Gaussian family $\left\{\mathcal{I}\left(h_{\mu_\epsilon^x}\right) : x \in \mathbb{R}^4, \epsilon > 0\right\}$ to construct a random measure on $\mathbb{R}^4$ which "formally" can be written as "$m(dx) = e^{2\gamma\theta(x)}dx$" where $\theta \in \Theta$ is chosen under the distribution of $\mathcal{W}$, $\gamma \geq 0$ is a constant and dx is the Lebesgue measure on $\mathbb{R}^4$. Recall that at every $x \in \mathbb{R}^4$, μ_ϵ^x tends to $2\delta_x$ as $\epsilon \downarrow 0$ in the sense of tempered distribution, so we can take the value of the random variable $\theta \mapsto \frac{1}{2}\mathcal{I}\left(h_{\mu_\epsilon^x}\right)(\theta)$ as the approximation for "$\theta(x)$" as $\epsilon \downarrow 0$. Furthermore, Corollary 3 certainly guarantees that with ϵ fixed, one can always make the mapping

$$(x,\theta) \in \mathbb{R}^4 \times \Theta \mapsto \mathcal{I}\left(h_{\mu_\epsilon^x}\right)(\theta) \in \mathbb{R}$$

measurable with respect to $\mathfrak{B}_{\mathbb{R}^4} \times \mathfrak{B}_\Theta$. In addition, we can assume for every $\theta \in \Theta$, the function

$$x \in \mathbb{R}^4 \mapsto E_\epsilon^\theta(x) \equiv \exp\left(\gamma\mathcal{I}\left(h_{\mu_\epsilon^x}\right)(\theta) - \frac{\gamma^2}{2}G(\epsilon)\right)$$

is positive and continuous, and hence if $m_\epsilon^\theta(dx) \equiv E_\epsilon^\theta(x)\,dx$, then $m_\epsilon^\theta(dx)$ is a positive regular and $\sigma-$finite Borel measure on $\mathbb{R}^4$. Moreover, given any Borel $B \subseteq \mathbb{R}^4$, the mapping

$$\theta \in \Theta \mapsto m_\epsilon^\theta(B) = \int_B E_\epsilon^\theta(x)\,dx \in \mathbb{R}$$

is also non-negative and measurable. Hence by Tonelli's Theorem and the fact that $\mathbb{E}^{\mathcal{W}}\left[E_\epsilon^\theta(x)\right] = 1$ for every $x \in \mathbb{R}^4$ and $\epsilon > 0$,

$$\mathbb{E}^{\mathcal{W}}\left[m_\epsilon^\theta(B)\right] = \int_B \mathbb{E}^{\mathcal{W}}\left[E_\epsilon^\theta(x)\right]dx = \mathrm{vol}(B).$$

Since $m_\epsilon^\theta(dx)$ is simply the Lebesgue measure on $\mathbb{R}^4$ when $\gamma = 0$, from now on we will only consider the case when $\gamma > 0$. We want to study the convergence of $m_\epsilon^\theta(dx)$ as $\epsilon \downarrow 0$. In order to have the desired convergence, we only consider ϵ taking values along a sequence $\{\epsilon_n \equiv \epsilon_0^n : n \geq 1\}$ for some fixed $\epsilon_0 \in (0,1)$. Without loss of generality, we will assume $m_{\epsilon_n}^\theta(dx)$ is well defined as above for all $n \geq 1$ and every $\theta \in \Theta$. For the sake of convenience in later discussions, we will abuse the

notation by identifying "$m^\theta_{\epsilon_0}(dx)$" with 0. We want to show that as $n\to\infty$, almost surely the sequence $\left\{m^\theta_{\epsilon_n}(dx)\right\}$ converges weakly to a limit measure $m^\theta(dx)$, written as $m^\theta_{\epsilon_n}(dx)\rightharpoonup m^\theta(dx)$, in the sense that $\int_{\mathbb{R}^4} f(x)\,m^\theta_{\epsilon_n}(dx)$ converges to $\int_{\mathbb{R}^4} f(x)\,m^\theta(dx)$ for every $f\in C_c\left(\mathbb{R}^4\right)$ which is the space of continuous and compactly supported functions on $\mathbb{R}^4$. To reach this goal, it suffices to show the convergence of $\int_{\mathbb{R}^4} f(x)\,m^\theta_{\epsilon_n}(dx)$ when f is any continuous function on Γ for any given compact set $\Gamma\subseteq\mathbb{R}^4$. In fact, we have the following result that holds for more general f so long as f is bounded and measurable on Γ.

Lemma 4. *Assume $0<\gamma^2<2\pi^2$ and $\Gamma\subseteq\mathbb{R}^4$ is compact. There exists a square integrable random variable $\theta\in\Theta\mapsto\overline{m^\theta}(\Gamma)\in\mathbb{R}^+$ such that*

$$\sum_{n=0}^{N-1}\left|m^\theta_{\epsilon_{n+1}}(\Gamma)-m^\theta_{\epsilon_n}(\Gamma)\right| \text{ converges to } \overline{m^\theta}(\Gamma) \text{ as } N\to\infty$$

almost surely as well as in $L^2(\mathcal{W})$.

Similarly, for every bounded measurable function f with $supp(f)\subseteq\Gamma$, there exists $M^\theta(f)\in L^2(\mathcal{W})$ such that

$$M^\theta_{\epsilon_n}(f)\equiv\int_{\mathbb{R}^4} f(x)\,m^\theta_{\epsilon_n}(dx) \text{ converges to } M^\theta(f) \text{ as } n\to\infty$$

almost surely as well as in $L^2(\mathcal{W})$, and $\left|M^\theta(f)\right|\le\overline{m^\theta}(\Gamma)\,\|f\|_u$ almost surely.

Proof. To prove the first statement, we rewrite $\left(m^\theta_{\epsilon_{n+1}}(\Gamma)-m^\theta_{\epsilon_n}(\Gamma)\right)^2$, $n\ge1$, as the following double integral:

$$\iint_{\Gamma^2}\left[E^\theta_{\epsilon_{n+1}}(x)\,E^\theta_{\epsilon_{n+1}}(y)+E^\theta_{\epsilon_n}(x)\,E^\theta_{\epsilon_n}(y)-2E^\theta_{\epsilon_{n+1}}(x)\,E^\theta_{\epsilon_n}(y)\right]dxdy.$$

By Tonelli's Theorem, the $\mathcal{W}$-expectation of above equals

$$\iint_{\Gamma^2}\left\{e^{\gamma^2\mathbb{E}\left[\mathcal{I}\left(h_{\mu^x_{\epsilon_{n+1}}}\right)\mathcal{I}\left(h_{\mu^y_{\epsilon_{n+1}}}\right)\right]}+e^{\gamma^2\mathbb{E}\left[\mathcal{I}\left(h_{\mu^x_{\epsilon_n}}\right)\mathcal{I}\left(h_{\mu^y_{\epsilon_n}}\right)\right]}-2e^{\gamma^2\mathbb{E}\left[\mathcal{I}\left(h_{\mu^x_{\epsilon_{n+1}}}\right)\mathcal{I}\left(h_{\mu^y_{\epsilon_n}}\right)\right]}\right\}dxdy.$$

We split this integral by dividing the domain into two parts:

$$\iint_{|x-y|>2\epsilon_n} \text{ and } \iint_{0\le|x-y|\le2\epsilon_n}.$$

The formula (2.10) implies the integrand is always zero in the designated domain of the first part. As for the second part, the integrand is always bounded by $2e^{\gamma^2G(\epsilon_{n+1})}$ while the volume of the integral domain is bounded by $C\epsilon_n^4$ for some constant[4] C. Together with the observations made in Section 2 about the asymptotics of G, one finds that

$$\mathbb{E}^{\mathcal{W}}\left[\left|m^\theta_{\epsilon_{n+1}}(\Gamma)-m^\theta_{\epsilon_n}(\Gamma)\right|^2\right]\le Ce^{-\left(8\pi^2-\gamma^2\right)G(\epsilon_n)}. \tag{3.1}$$

[4]Throughout this section, "C" denotes a positive finite constant that is universal in ϵ_n. The value of C may change from line to line.

The square root of the right hand side of (3.1) is summable in $n \geq 1$ and meanwhile $m^{\theta}_{\epsilon_1}(\Gamma)$ is clearly square integrable, so

$$\overline{m^{\theta}}(\Gamma) \equiv \sum_{n=0}^{\infty} \left| m^{\theta}_{\epsilon_{n+1}}(\Gamma) - m^{\theta}_{\epsilon_n}(\Gamma) \right| \tag{3.2}$$

is square integrable and the convergence takes place in $L^2(\mathcal{W})$. Furthermore, the series $\sum_{n=0}^{N} \left| m^{\theta}_{\epsilon_{n+1}}(\Gamma) - m^{\theta}_{\epsilon_n}(\Gamma) \right|$ also converges to $\overline{m^{\theta}}(\Gamma)$ almost surely along a subsequence, but since the series is monotonic in N, it must converge to $\overline{m^{\theta}}(\Gamma)$ almost surely along the full sequence.

The second statement of the lemma follows from the same arguments. In fact, given a bounded measurable function f with $\mathrm{supp}(f) \subseteq \Gamma$, if one replaces $m^{\theta}_{\epsilon_n}(\Gamma)$ by $M^{\theta}_{\epsilon_n}(f)$ in every step of the proof above, one can see that $\sum_{n=0}^{\infty} \left| M^{\theta}_{\epsilon_{n+1}}(f) - M^{\theta}_{\epsilon_n}(f) \right|$ is also square integrable. The rest of the proof is straightforward. □

Theorem 5. *Assume $0 < \gamma^2 < 2\pi^2$. For almost every $\theta \in \Theta$, there exists a non-negative regular $\sigma-$finite Borel measure $m^{\theta}(dx)$ on $\mathbb{R}^4$ such that*

$$m^{\theta}_{\epsilon_n}(dx) \rightharpoonup m^{\theta}(dx) \ \text{ as } n \to \infty,$$

and for every compact set $\Gamma \subseteq \mathbb{R}^4$, $\left\|m^{\theta}\right\|_{var,\Gamma} \leq \overline{m^{\theta}}(\Gamma)$ where $\left\|m^{\theta}\right\|_{var,\Gamma}$ is the total variation of $m^{\theta}(dx)$ over Γ and $\overline{m^{\theta}}(\Gamma)$ is as defined in (3.2).

In particular, for every $f \in C_c\left(\mathbb{R}^4\right)$,

$$\int_{\mathbb{R}^4} f(x)\, m^{\theta}_{\epsilon_n}(dx) \ \text{ converges to } \int_{\mathbb{R}^4} f(x)\, m^{\theta}(dx) \ \text{ as } n \to \infty$$

almost surely as well as in $L^2(\mathcal{W})$.

Proof. Clearly we only need to prove the first statement of the theorem, because assuming $m^{\theta}_{\epsilon_n}(dx) \rightharpoonup m^{\theta}(dx)$ almost surely, the second statement is simply repeating the second result in Lemma 4 with $M^{\theta}(f) = \int_{\mathbb{R}^4} f(x)\, m^{\theta}(dx)$ for $f \in C_c\left(\mathbb{R}^4\right)$. As we mentioned earlier, to obtain the limit measure $m^{\theta}(dx)$, it suffices to show the convergence of $m^{\theta}_{\epsilon_n}(dx)$ on any compact set $\Gamma \subseteq \mathbb{R}^4$. We will achieve this goal via the Riesz representation theorem. We have already seen from the second part of Lemma 4 that, if denote $M^{\theta}_{\epsilon_n}(f) \equiv \int_{\mathbb{R}^4} f(x)\, m^{\theta}_{\epsilon_n}(dx)$ for every $n \geq 1$ and every bounded measurable function f supported on Γ, then

$$M^{\theta}(f) \equiv \lim_{n\to\infty} M^{\theta}_{\epsilon_n}(f) \ \text{ exists and } \left|M^{\theta}(f)\right| \leq \|f\|_u\, \overline{m^{\theta}}(\Gamma) < \infty \tag{3.3}$$

with probability one. However, to get the almost sure existence of $m^{\theta}(dx)$, we need to argue that with probability one, the statement above holds simultaneously for all functions f from a "large enough" class. To this end, we make use of the separability of the Banach space $C(\Gamma)$ (when equipped with the uniform norm $\|\cdot\|_u$). Namely, we can choose a countable sequence $\{f_k : k \geq 1\}$ which is a dense subset of $C(\Gamma)$, so almost surely the statement (3.3) holds[5] simultaneously for all f_k, $k \geq 1$.

Now let's focus on $\theta \in \Theta$ such that (3.3) holds for all f_k, $k \geq 1$. Given a general $f \in C(\Gamma)$, let $\left\{f_{k_j} : j \geq 1\right\}$ be a subsequence such that $f_{k_j} \to f$ in $\|\cdot\|_u$ as $j \to \infty$,

[5] In this discussion, we will simply assume $f \equiv 0$ outside Γ for every $f \in C(\Gamma)$, so $M^{\theta}(f)$ and $M^{\theta}_{\epsilon_n}(f)$ are still well defined.

then for every $l,n\geq 1$,

$$\begin{aligned}\left|M^{\theta}_{\epsilon_l}(f)-M^{\theta}_{\epsilon_n}(f)\right| &\leq \left|M^{\theta}_{\epsilon_l}(f)-M^{\theta}_{\epsilon_l}\left(f_{k_j}\right)\right|+\left|M^{\theta}_{\epsilon_l}\left(f_{k_j}\right)-M^{\theta}_{\epsilon_n}\left(f_{k_j}\right)\right| \\ &\qquad +\left|M^{\theta}_{\epsilon_n}\left(f_{k_j}\right)-M^{\theta}_{\epsilon_n}(f)\right| \\ &\leq 2\overline{m^{\theta}}(\Gamma)\left\|f-f_{k_j}\right\|_u+\left|M^{\theta}_{\epsilon_l}\left(f_{k_j}\right)-M^{\theta}_{\epsilon_n}\left(f_{k_j}\right)\right|.\end{aligned}$$

Obviously $\left\{M^{\theta}_{\epsilon_n}(f):n\geq 1\right\}$ forms a Cauchy sequence which immediately implies that (3.3) also holds for f and in addition $M^{\theta}(f)=\lim_{j\to\infty}M^{\theta}\left(f_{k_j}\right)$. Therefore, for almost every $\theta\in\Theta$, $f\mapsto M^{\theta}(f)$ is a linear and bounded functional on $C(\Gamma)$, which, by the Riesz representation theorem, gives rise to a unique regular Borel measure $m^{\theta}(dx)$ on Γ such that $M^{\theta}(f)=\int_{\Gamma}f(x)\,m^{\theta}(dx)$ for all $f\in C(\Gamma)$ and the total variation of $m^{\theta}(dx)$ is equal to the operator norm of M^{θ} which is bounded by $\overline{m^{\theta}}(\Gamma)$. It's also clear that $m^{\theta}(dx)$ is non-negative and $m^{\theta}_{\epsilon_n}(dx)\rightharpoonup m^{\theta}(dx)$. □

Compared with the second statement in Lemma 4, the second statement in Theorem 5 seems "weaker" since we have restricted ourselves to $f\in C_c\left(\mathbb{R}^4\right)$. However, we point out that the same statement, i.e., $M^{\theta}_{\epsilon_n}(f)$ converges to $\int_{\mathbb{R}^4}f(x)\,m^{\theta}(dx)$ both almost surely and in $L^2(\mathcal{W})$, is no longer true if f is only assumed to be a bounded measurable function with compact support. The reason is the following: for such a function f, although the existence of $M^{\theta}(f)=\lim_{n\to\infty}M^{\theta}_{\epsilon_n}(f)$ is guaranteed by Lemma 4, in general one cannot draw any conclusion on the relation between $M^{\theta}(f)$ and $\int_{\mathbb{R}^4}f(x)\,m^{\theta}(dx)$, because $m^{\theta}_{\epsilon_n}(dx)$ only converges to $m^{\theta}(dx)$ weakly and one does not have control over $\left\|m^{\theta}_{\epsilon_n}-m^{\theta}\right\|_{\mathrm{var,supp}(f)}$. However, under some circumstances, we can derive a relation between the two random variables $M^{\theta}(f)$ and $\int_{\mathbb{R}^4}f(x)\,m^{\theta}(dx)$. For example, if $f=\chi_A$ is the indicator function of a bounded open set $A\subseteq\mathbb{R}^4$, then the weak convergence result implies $m^{\theta}(A)\leq\lim_{n\to\infty}m^{\theta}_{\epsilon_n}(A)$ almost surely. Meanwhile, the $L^2(\mathcal{W})$ convergence of $m^{\theta}_{\epsilon_n}(A)$ certainly leads to

$$\mathbb{E}^{\mathcal{W}}\left[\lim_{n\to\infty}m^{\theta}_{\epsilon_n}(A)\right]=\lim_{n\to\infty}\mathbb{E}^{\mathcal{W}}\left[m^{\theta}_{\epsilon_n}(A)\right]=\mathrm{vol}(A);$$

on the other hand, let $\{f_l:l\geq 1\}\subseteq C_c\left(\mathbb{R}^4\right)$ be a sequence such that $0\leq f_l\nearrow\chi_A$ as $l\to\infty$, then by the monotone convergence theorem and the second statement in Theorem 5,

$$\mathbb{E}^{\mathcal{W}}\left[m^{\theta}(A)\right]=\lim_{l\to\infty}\mathbb{E}^{\mathcal{W}}\left[\int_{\mathbb{R}^4}f_l(x)\,m^{\theta}(dx)\right]=\lim_{l\to\infty}\int_{\mathbb{R}^4}f_l(x)\,dx=\mathrm{vol}(A).$$

This can only be possible if $m^{\theta}(A)=\lim_{n\to\infty}m^{\theta}_{\epsilon_n}(A)$ almost surely. More generally ([S11], §9.1), if f is bounded and upper semicontinuous (or lower semicontinuous or $m^{\theta}-$almost surely continuous), then it follows from a similar argument that $M^{\theta}(f)=\lim_{n\to\infty}M^{\theta}_{\epsilon_n}(f)$ almost surely, so the second statement in Theorem 5 also holds for f.

The fact as stated above that $\mathbb{E}^{\mathcal{W}}\left[m^{\theta}(A)\right]=\mathrm{vol}(A)$ for every bounded open set guarantees that the limit measure $m^{\theta}(dx)$ cannot be almost everywhere trivial, i.e., $\mathcal{W}\left(m^{\theta}(dx)=0\right)<1$. In fact, we will prove later (in Lemma 10) that $\mathcal{W}\left(m^{\theta}(dx)=0\right)=0$, so $m^{\theta}(dx)$ is almost surely a positive measure. On the other hand, the following simple observation shows that $m^{\theta}(dx)$ almost surely does not assign positive mass to any given point. To see this, recall the assumption

$0 < \gamma^2 < 2\pi^2$ and the fact that $G(r)$ is asymptotic to $-\frac{1}{2\pi^2}\log r$ when r is small. Then it is an easy matter to check that for any fixed $x \in \mathbb{R}^4$,

$$\mathbb{E}^{\mathcal{W}}\left[\limsup_{n\to\infty} e^{4\gamma^2 G(\epsilon_n)} m^\theta\left(\overline{B_{\epsilon_n}(x)}\right)\right] = 0.$$

Therefore, if denote

$$\Theta_x \equiv \left\{\theta \in \Theta : \lim_{n\to\infty} e^{4\gamma^2 G(\epsilon_n)} m^\theta\left(\overline{B_{\epsilon_n}(x)}\right) = 0\right\}, \tag{3.4}$$

then Θ_x is clearly a measurable subset of Θ and $\mathcal{W}(\Theta_x) = 1$.

We will close this section by a remark about the condition of the constant[6] γ. Readers may have noticed that the constraint $0 < \gamma^2 < 2\pi^2$ in Lemma 4 and Theorem 5 is more than what the proofs require. However, one reason of having this condition on γ is that it guarantees the proof of Lemma 4 being correct even if one replaces γ by 2γ. In other words, if we denote

$$m_{\epsilon_n}^{\theta,2\gamma}(dx) \equiv e^{2\gamma\mathcal{I}\left(h_{\mu^x_{\epsilon_n}}\right)(\theta) - 2\gamma^2 G(\epsilon_n)}dx$$

and define $\overline{m^{\theta,2\gamma}}(\Gamma)$ similarly using $m_{\epsilon_n}^{\theta,2\gamma}(\Gamma)$ for any compact set $\Gamma \subseteq \mathbb{R}^4$, then $\overline{m^{\theta,2\gamma}}(\Gamma)$ is also square integrable and in particular $\overline{m^{\theta,2\gamma}}(\Gamma)$ is almost surely finite. Some proofs in Section 5 make use of this consideration and hence the condition $0 < \gamma^2 < 2\pi^2$ becomes necessary there. We will remind readers when it comes to those situations.

4. KPZ Relation

Throughout later discussions, we will always assume $0 < \gamma^2 < 2\pi^2$ and for every $\theta \in \Theta$, $m^\theta(dx)$ is a non-negative regular and $\sigma-$finite Borel measure on $\mathbb{R}^4$ and $m^\theta_{\epsilon_n}(dx) \rightharpoonup m^\theta(dx)$ (otherwise one simply assigns $m^\theta_{\epsilon_n}(dx) = m^\theta(dx) = dx$ for all $n \geq 1$ on a measurable null set of Θ). In this section, we would like to establish a KPZ relation between the Euclidean scaling exponent of a bounded (fractal) Borel set on $\mathbb{R}^4$ and its counterpart under the random measure $m^\theta(dx)$. We first recall from [DS11] some definitions. Given a bounded Borel $D \subseteq \mathbb{R}^4$, the constant $\kappa \in [0,1]$ is called the Euclidean *scaling exponent* of D if

$$\lim_{\lambda\downarrow 0}\frac{\log \mathrm{vol}(D_\lambda)}{\log \lambda^4} = \kappa,$$

where $\lambda > 0$ and $D_\lambda \equiv \cup_{x\in D} B_\lambda(x)$ is the canonical $\lambda-$neighborhood of D. In the random measure setting, for every $\Lambda > 0$, we first consider the mapping from $\mathbb{R}^4 \times \Theta$ to $[0,\infty]$:

$$(x,\theta) \mapsto r_\Lambda(x,\theta) \equiv \begin{cases} \sup\left\{r > 0 : m^\theta(B_r(x)) \leq \Lambda\right\} & \text{if } \theta \in \Theta_x, \\ 0 & \text{otherwise,}\end{cases} \tag{4.1}$$

where Θ_x is as determined in (3.4), then define the *isothermal* $\Lambda-$*neighborhood* of D by

$$D^{\Lambda,\theta} \equiv \left\{x \in \mathbb{R}^4 : \begin{array}{l} \text{either } r_\Lambda(x,\theta) > 0 \text{ and } \mathrm{dist}(x,D) < r_\Lambda(x,\theta) \\ \text{or } r_\Lambda(x,\theta) = 0 \text{ and } x \in D \end{array}\right\}. \tag{4.2}$$

[6]In this article, we don't have particular emphasis on the potential physics meaning of γ.

Intuitively, $B_{r_\Lambda(x,\theta)}(x)$ is the "largest" ball (in radius) centered at x with volume Λ under the random measure $m^\theta(dx)$, and $D^{\Lambda,\theta}$ is the "neighborhood" obtained by covering D with all such balls with equal volume. If there exists constant $K \in [0,1]$ such that

$$\lim_{\Lambda \downarrow 0} \frac{\log \mathbb{E}^{\mathcal{W}}\left[m^\theta\left(D^{\Lambda,\theta}\right)\right]}{\log \Lambda} = K, \tag{4.3}$$

then K is called the *quantum scaling exponent* of D. K can be viewed as the "expected" scaling exponent and as such the counterpart of κ under the random measure $m^\theta(dx)$. In two dimensions, κ and K satisfy the so-called KPZ relation which is a quadratic relation. Our goal in this section is to extend such a relation to the four dimensional setting.

However, we haven't yet justified the "meaning" of the notations in (4.3). First we have to check that $D^{\Lambda,\theta}$ is a Borel set in $\mathbb{R}^4$ for every $\theta \in \Theta$, which requires us to verify the measurability of $(x,\theta) \mapsto r_\Lambda(x,\theta)$ with respect to $\mathfrak{B}_{\mathbb{R}^4} \times \mathfrak{B}_\Theta$. To this end, we observe that $(x,\theta) \mapsto m^\theta(B_r(x))$ is measurable for every $r > 0$ because there certainly exists continuous mapping $x \in \mathbb{R}^4 \mapsto \rho_l^x \in C_c\left(\mathbb{R}^4\right)$ for every $l \geq 1$ with $0 \leq \rho_l^x \nearrow \chi_{B_r(x)}$ and hence $M^\theta\left(\rho_l^x\right) \nearrow m^\theta(B_r(x))$ as $l \to \infty$ for every (x,θ). From this we conclude that $\{(x,\theta) : \theta \in \Theta_x\}$ is a Borel set in $\mathbb{R}^4 \times \Theta$, and then the measurability of $r_\Lambda(x,\theta)$ follows simply by identifying the set $\{(x,\theta) : 0 < r_\Lambda(x,\theta) \leq a\}$ with $\left\{(x,\theta) : \theta \in \Theta_x, m^\theta(B_a(x)) \geq \Lambda\right\}$ for every $a > 0$. Therefore, for every θ, $D^{\Lambda,\theta}$ is a Borel set in $\mathbb{R}^4$ since it is the $\theta-$section of the following Borel set in $\mathbb{R}^4 \times \Theta$:

$$(D \times \Theta) \cup \{(x,\theta) : r_\Lambda(x,\theta) > 0,\ \operatorname{dist}(x,D) < r_\Lambda(x,\theta)\}.$$

Next, we need to show that $\theta \mapsto m^\theta\left(D^{\Lambda,\theta}\right)$ is measurable with respect to $\mathfrak{B}_\Theta$. More generally, if $C \in \mathfrak{B}_{\mathbb{R}^4} \times \mathfrak{B}_\Theta$ and $C^\theta \equiv \left\{x \in \mathbb{R}^4 : (x,\theta) \in C\right\}$ is the $\theta-$section of C, we claim that $\theta \mapsto m^\theta\left(C^\theta\right)$ is measurable with respect to $\mathfrak{B}_\Theta$. To see this, denote $C_N^\theta \equiv C^\theta \cap B_N(0)$ for every $N \geq 1$ and choose a sequence of mollifiers $\{\varphi_k \in [0,1] : k \geq 1\} \subseteq C_c\left(\mathbb{R}^4\right)$ such that $g_k^\theta \equiv \varphi_k \star \chi_{C_N^\theta}$ converges to $\chi_{C_N^\theta}$ under $\|\cdot\|_u$ as $k \to \infty$. Moreover, for every $k \geq 1$, since $C \in \mathfrak{B}_{\mathbb{R}^4} \times \mathfrak{B}_\Theta$,

$$(x,\theta) \mapsto g_k^\theta(x) = \int_{(y,\theta) \in C, |y| < N} \varphi_k(x-y)\, dy$$

is also measurable with respect to $\mathfrak{B}_{\mathbb{R}^4} \times \mathfrak{B}_\Theta$. Thus, because $g_k^\theta \in C_c\left(\mathbb{R}^4\right)$ and $\left\|g_k^\theta\right\|_u \leq 1$ for every $k \geq 1$ and $m_{\epsilon_n}^\theta(dx) \rightharpoonup m^\theta(dx)$,

$$m^\theta\left(C^\theta\right) = \lim_{N \to \infty} \lim_{k \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^4} g_k^\theta(x)\, E_{\epsilon_n}^\theta(x)\, dx,$$

which is a measurable function in θ.

At this point, one can already tell that it is convenient to consider the spatial variable x and the GFF θ at the same time. In particular, if $\mathcal{M}(dxd\theta) \equiv m^\theta(dx)\, \mathcal{W}(d\theta)$, then it's clear from the preceding that $\mathcal{M}(dxd\theta)$ is a non-negative $\sigma-$finite Borel measure on the product space $\mathbb{R}^4 \times \Theta$. To connect κ with K, it is natural to investigate the distribution of $r_\Lambda(x,\theta)$ under the $\mathcal{M}(dxd\theta)$. To get started, we will first look at the distribution of $m^\theta(B_r(x))$ under $\mathcal{M}(dxd\theta)$ for any given $r > 0$. The following lemma says that we can change our "perspective" by first choosing a "base" point $x \in \mathbb{R}^4$ and then examine the distribution of $m^\theta(B_r(x))$

at x. This is realized by a procedure of exchanging the order of integration. The proof of Lemma 6 is given in Section 5.

Lemma 6. *For every $x \in \mathbb{R}^4$, let Θ_x be the measurable subset of Θ as determined in (3.4), and define*

$$\theta \mapsto \hat{m}^{\theta,x}(dy) \equiv \begin{cases} \exp\left(\frac{\gamma^2}{2\pi^2} K_0(|x-y|)\right) m^\theta(dy) & \text{if } \theta \in \Theta_x, \\ m^\theta(dy) & \text{otherwise.} \end{cases} \tag{4.4}$$

Then almost surely $\hat{m}^{\theta,x}(dy)$ is also a non-negative regular and $\sigma-$finite Borel measure on $\mathbb{R}^4$.

Moreover, for every $r > 0$, compact set $\Gamma \subseteq \mathbb{R}^4$ and $F \in C_0\left(\mathbb{R}^4 \times [0,\infty)\right)$,

$$\int_\Theta \int_\Gamma F\left(x, m^\theta(B_r(x))\right) m^\theta(dx)\, \mathcal{W}(d\theta) = \int_\Gamma \int_\Theta F\left(x, \hat{m}^{\theta,x}(B_r(x))\right) \mathcal{W}(d\theta)\, dx. \tag{4.5}$$

In particular, this implies that for every $r > 0$, the joint distribution of $\left(x, m^\theta(B_r(x))\right)$ under $\mathcal{M}(dxd\theta)$ is the same as the distribution of $\left(x, \hat{m}^{\theta,x}(B_r(x))\right)$ under $\mathcal{W}(d\theta)\,dx$, whose marginal distribution on Θ at x is independent of x.

Based on Lemma 6, instead of $m^\theta(B_r(x))$ under $\mathcal{M}(dxd\theta)$, we may as well study the distribution of $\hat{m}^{\theta,x}(B_r(x))$ under $\mathcal{W}(d\theta)\,dx$. Similarly, to understand $r_\Lambda(x,\theta)$ under $\mathcal{M}(dxd\theta)$, we only need to look at the random variable given by

$$(x,\theta) \mapsto \hat{r}_\Lambda(x,\theta) \equiv \begin{cases} \sup\left\{r > 0 : \hat{m}^{\theta,x}(B_r(x)) \le \Lambda\right\} & \text{if } \theta \in \Theta_x, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

under $\mathcal{W}(d\theta)\,dx$, whose marginal distribution on Θ at x is again independent of x.

To proceed from here, we will follow the same strategy as in [DS11]. For the sake of completeness, we will still present the main steps here. For every $r > 0$ and $\Lambda > 0$, since the distribution of $\hat{m}^{\theta,x}(B_r(x))$ and $\hat{r}_\Lambda(x,\theta)$ under $\mathcal{W}$ does not depend on x, we can assume x is the origin and simplify the notation by denoting $B_r \equiv B_r(0)$, $\hat{r}_\Lambda(\theta) \equiv \hat{r}_\Lambda(0,\theta)$ and $\hat{m}^\theta(B_r) \equiv \hat{m}^{\theta,0}(B_r(0))$. We want to find an "approximation" for $\hat{m}^\theta(B_r)$ by conditioning on the value of the GFF restricted to the "boundary" of B_r. To be precise, recall from the definition (2.11) that if $r(t) \equiv G^{-1}(t + G(R))$ with $t \ge 0$, then $X_t = \mathcal{I}\left(h_{\mu^0_{r(t)}}\right) - \mathcal{I}\left(h_{\mu^0_R}\right)$ has the same distribution of a standard Brownian motion. We want to investigate the conditional expectation of $\hat{m}^\theta\left(B_{r(t)}\right)$ given X_t. To do this, we need to relate $\hat{m}^\theta\left(B_{r(t)}\right)$ to the approximating measures $m^\theta_{\epsilon_n}(dx)$, which requires us to overcome the singularity of $e^{\frac{\gamma^2}{2\pi^2}K_0(|\cdot|)}$ at the origin. To this end, assume $\{f_l : l \ge 1\} \subseteq C_c\left(\overline{B_R}\right)$ is a sequence with $0 \le f_l \nearrow \chi_{B_{r(t)}}$ as $l \to \infty$, then[7]

$$C_c\left(\overline{B_R}\right) \ni d_l(\cdot) \equiv f_l(\cdot)\, e^{\frac{\gamma^2}{2\pi^2}(K_0(|\cdot|)\wedge l)} \nearrow \chi_{B_{r(t)}}(\cdot)\, e^{\frac{\gamma^2}{2\pi^2}K_0(|\cdot|)}.$$

Therefore, one can apply the convergence results in Theorem 5 to d_l for every $l \ge 1$. Together with the monotone convergence theorem, one sees that for every $t \ge 0$,

$$\mathbb{E}^{\mathcal{W}}\left[\hat{m}^\theta\left(B_{r(t)}\right) | X_t\right] = \lim_{l\to\infty} \liminf_{n\to\infty} \mathbb{E}^{\mathcal{W}}\left[M^\theta_{\epsilon_n}(d_l) | X_t\right]. \tag{4.7}$$

[7]The notation "$\alpha \vee \beta$" denotes "$\max\{\alpha,\beta\}$" and "$\alpha \wedge \beta$" denotes "$\min\{\alpha,\beta\}$".

Given the order of taking limits in the right hand side of (4.7), for every $l \geq 1$ and eventually all large n, we have $\overline{B_{\epsilon_n}(y)} \subseteq B_{r(t)} \subseteq B_R$ for every $y \in \operatorname{supp}(d_l)$. Thus by a simple exercise on conditional expectations of Gaussian random variables along with (2.9), we can derive from (4.7) that

$$\begin{aligned} & \mathbb{E}^{\mathcal{W}}\left[\hat{m}^\theta\left(B_{r(t)}\right)|X_t\right] \\ = & \int_{B_{r(t)}} e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)} \exp\left[\left(I_0(|y|) - I_2(|y|)P(t)\right)\gamma X_t\right] \\ & \cdot \exp\left[-\frac{\gamma^2 t}{2}\left(I_0(|y|) - I_2(|y|)P(t)\right)^2\right] dy, \end{aligned} \tag{4.8}$$

where $P(t) \equiv \left(4\pi^2 t\right)^{-1}\left[\left(I_1^2 - I_0 I_2\right)^{-1}(r(t)) - \left(I_1^2 - I_0 I_2\right)^{-1}(R)\right]$.

If one carefully examines the asymptotics near the origin of the Bessel functions involved, one realizes that (4.8) suggests the conditional expectation of $\hat{m}^\theta\left(B_{r(t)}\right)$ given X_t, when t is large, is "approximately"

$$\hat{m}^{\theta *}\left(B_{r(t)}\right) \equiv \exp\left(\gamma X_t - \left(8\pi^2 - \frac{\gamma^2}{2}\right)t\right). \tag{4.9}$$

For the moment we will "pretend" $\hat{m}^\theta\left(B_{r(t)}\right)$ is just $\hat{m}^{\theta *}\left(B_{r(t)}\right)$ and formulate the KPZ relation under this circumstance.

For every $\Lambda > 0$, we define the stopping time:

$$T_\Lambda^* \equiv \inf\left\{t \geq 0 : \hat{m}^{\theta *}\left(B_{r(t)}\right) = \exp\left(\gamma X_t - \left(8\pi^2 - \frac{\gamma^2}{2}\right)t\right) \leq \Lambda\right\}. \tag{4.10}$$

The distribution of T_Λ^* can be completely determined by a martingale argument, which is straightforward but worth repeating. Namely, for every $s \leq 0$, by Doob's stopping time theorem, $\left\{\exp\left[sX_{t\wedge T_\Lambda^*} - \frac{s^2}{2}\left(t \wedge T_\Lambda^*\right)\right] : t \geq 0\right\}$ is a uniformly bounded martingale. Furthermore, the continuity of Brownian motion implies that

$$X_{T_\Lambda^*} = \frac{\log \Lambda}{\gamma} + \left(8\pi^2 - \frac{\gamma^2}{2}\right)\frac{T_\Lambda^*}{\gamma}.$$

Therefore, the fact that the expectation of the martingale at $t = 0$ is equal to that at $t = T_\Lambda^*$ leads to the formula of the moment generating function of T_Λ^*:

$$\mathbb{E}^{\mathcal{W}}\left[\exp\left(-\frac{\gamma s^2 - 2s\left(8\pi^2 - \frac{\gamma^2}{2}\right)}{2\gamma}T_\Lambda^*\right)\right] = \Lambda^{-s/\gamma}. \tag{4.11}$$

From here we can derive our first version of the KPZ relation which is easy but revealing.

Lemma 7. *Assume $D \subseteq \mathbb{R}^4$ is a bounded Borel set with Euclidean scaling exponent $\kappa \in [0,1]$, i.e.,*

$$\lim_{\lambda \downarrow 0} \frac{\log Vol(D_\lambda)}{\log \lambda^4} = \kappa. \tag{4.12}$$

For every $\Lambda > 0$, let T_Λ^* be as in (4.10) and define the random "radius":

$$\theta \mapsto r_\Lambda^*(\theta) \equiv G^{-1}\left(T_\Lambda^*(\theta) + G(R)\right),$$

and the random "neighborhood":

$$\theta \mapsto D^{\Lambda *, \theta} \equiv \cup_{x \in D} B_{r_\Lambda^*(\theta)}(x).$$

Then, we have

$$\lim_{\Lambda\downarrow 0}\frac{\log\mathbb{E}^{\mathcal{W}}\left[m^{\theta}\left(D^{\Lambda*,\theta}\right)\right]}{\log\Lambda}=\lim_{\Lambda\downarrow 0}\frac{\log\mathbb{E}^{\mathcal{W}}\left[\left(r_{\Lambda}^{*}\right)^{4\kappa}\right]}{\log\Lambda}=K \tag{4.13}$$

where $K\in[0,1]$ *is determined by the following quadratic relation with* κ*:*

$$\kappa=K\left(1-\frac{\gamma^2}{16\pi^2}\right)+\frac{\gamma^2}{16\pi^2}K^2. \tag{4.14}$$

Proof. Clearly in this setting, we want to cover D with open balls that all have the same "critical" radius determined by the stopping time associated with the martingale $\hat{m}^{\theta*}\left(B_{r(t)}\right)$ (defined in (4.9)). We don't have to worry about the measurability of $\theta\mapsto m^{\theta}\left(D^{\Lambda*,\theta}\right)$ because both $\theta\mapsto r_{\Lambda}^{*}(\theta)$ and $(\theta,r)\mapsto m^{\theta}(D_r)$ are measurable. Conditioning on $r_{\Lambda}^{*}=r\in(0,R]$, $D^{\Lambda*,\theta}$ is bounded and open, and the conditional expectation of $m^{\theta}\left(D^{\Lambda*,\theta}\right)$ is a multiple (reciprocal of the probability density function of r_{Λ}^{*} at r) of $\mathrm{vol}(D_r)$, which, according to (4.12), [8]"$\approx$" $r^{4\kappa}$ for every $r\in(0,R]$. This means $\mathbb{E}^{\mathcal{W}}\left[m^{\theta}\left(D^{\Lambda*,\theta}\right)\right]\approx\mathbb{E}^{\mathcal{W}}\left[\left(r_{\Lambda}^{*}\right)^{4\kappa}\right]$ and further $\approx\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T_{\Lambda}^{*}\right)\right]$. Given (4.11), clearly one wants to set $8\pi^2\kappa$ to be $\frac{s^2}{2}-\frac{s}{\gamma}\left(8\pi^2-\frac{\gamma^2}{2}\right)$ for some $s\in[-\gamma,0]$, in which case

$$\lim_{\Lambda\downarrow 0}\frac{\log\mathbb{E}^{\mathcal{W}}\left[m^{\theta}\left(D^{\Lambda*,\theta}\right)\right]}{\log\Lambda}=\lim_{\Lambda\downarrow 0}\frac{\log\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T_{\Lambda}^{*}\right)\right]}{\log\Lambda}=-\frac{s}{\gamma}.$$

The results in (4.13) and (4.14) follow immediately after setting $K\equiv-\frac{s}{\gamma}$. □

Next, we argue that $\hat{m}^{\theta*}(B_r)$ is indeed a "legitimate" approximation for $\hat{m}^{\theta}(B_r)$ in the sense that r_{Λ}^{*}, as defined in Lemma 7, approximates $\hat{r}_{\Lambda}:\theta\mapsto\hat{r}_{\Lambda}(\theta)$ when "compared" in the limit of the logarithm ratio.

Lemma 8. *Assume the pair* $(\kappa,K)\in[0,1]^2$ *satisfies the quadratic relation in (4.14). Then,*

$$\lim_{\Lambda\downarrow 0}\frac{\log\mathbb{E}^{\mathcal{W}}\left[\left(\hat{r}_{\Lambda}\right)^{4\kappa}\right]}{\log\Lambda}=K\ \textit{or equivalently}\ \lim_{\Lambda\downarrow 0}\frac{\log\mathbb{E}^{\mathcal{W}}\left[\left(\hat{r}_{\Lambda}\right)^{4\kappa}\right]}{\log\mathbb{E}^{\mathcal{W}}\left[\left(r_{\Lambda}^{*}\right)^{4\kappa}\right]}=1. \tag{4.15}$$

The proof of this lemma is given in Section 5. There we also prove a preliminary result (Lemma 10) which actually implies the almost sure non-triviality of the measure $\hat{m}^{\theta}(dx)$ as well as $m^{\theta}(dx)$. Most importantly, this lemma builds up the final passage to the KPZ relation for $m^{\theta}(dx)$, the "true" case in which we are interested. Again, we will only present the statement here and leave the proof to the next section.

Theorem 9. *Let* $D\subseteq\mathbb{R}^4$ *be a bounded Borel set with Euclidean scaling exponent* $\kappa\in[0,1]$*. Then* D *has quantum scaling exponent* $K\in[0,1]$ *as defined in (4.3), where* K *is related to* κ *by (4.14).*

[8]Throughout this article, the notation "$\approx$" means "bounded from above and below by a universal constant multiple of".

5. Proofs of Results in Section 4

We will now prove Lemma 6. The strategy is to relate $m^\theta(dx)$ to the approximating measures $m^\theta_{\epsilon_n}(dx)$ and recognize that, by the Cameron-Martin formula, the density of $m^\theta_{\epsilon_n}(dx)$, i.e., $E^\theta_{\epsilon_n}(x) = \exp\left(\gamma\mathcal{I}\left(h_{\mu^x_{\epsilon_n}}\right)(\theta) - \frac{\gamma^2}{2}G(\epsilon_n)\right)$, is just the Radon-Nikodym derivative with respect to $\mathcal{W}$ of the Gaussian measure induced by the translation $\theta \mapsto \theta + \gamma h_{\mu^x_{\epsilon_n}}$ under $\mathcal{W}$. Note that the constraint $0 < \gamma^2 < 2\pi^2$ becomes necessary in this proof.

Proof of Lemma 6: Let $\hat{m}^{\theta,x}(dy)$ be the measure as defined in (4.4). The claim that $\hat{m}^{\theta,x}(dy)$ is non-negative regular and $\sigma-$finite follows from the observation that $\exp\left(\frac{\gamma^2}{2\pi^2}K_0(|x-\cdot|)\right)$ is locally integrable with respect to $m^\theta(dx)$ if $\theta \in \Theta_x$. Without loss of generality, we will assume $x = 0$. The only possible problem comes from the singularity at 0. However, if we rewrite

$$\begin{aligned}\int_{B_{\epsilon_0}(0)} e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)} m^\theta(dy) &= \sum_{k=0}^\infty \int_{\epsilon_k \le |y| < \epsilon_{k-1}} e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)} m^\theta(dy) \\ &\le \sum_{k=0}^\infty e^{\frac{\gamma^2}{2\pi^2}K_0(\epsilon_k)} m^\theta\left(B_{\epsilon_{k-1}}(0)\right),\end{aligned}$$

then the criterion (3.4) for Θ_x guarantees that the series in the right hand side of above is convergent.

Now we move on to the second part of the lemma. Clearly both mappings

$$(x,\theta) \mapsto F\left(x, m^\theta(B_r(x))\right) \text{ and } (x,\theta) \mapsto F\left(x, \hat{m}^{\theta,x}(B_r(x))\right)$$

are measurable with respect to $\mathfrak{B}_{\mathbb{R}^4} \times \mathfrak{B}_\Theta$, so the two integrals in (4.5) are well defined and in fact finite. Choose a continuous mapping $x \in \Gamma \mapsto \rho^x \in C_0\left(\mathbb{R}^4\right)$ with $0 \le \rho^x < \chi_{B_r(x)}$. We first show (4.5) holds with $\chi_{B_r(x)}$ replaced by ρ^x. Namely, we claim that

$$\begin{aligned}&\int_\Theta \int_\Gamma F\left(x, M^\theta(\rho^x)\right) m^\theta(dx)\, \mathcal{W}(d\theta) = \\ &\qquad \int_\Gamma \int_\Theta F\left(x, M^\theta\left(\rho^x e^{\frac{\gamma^2}{2\pi^2}K_0(|x-\cdot|)}\right)\right) \mathcal{W}(d\theta)\, dx.\end{aligned} \tag{5.1}$$

We start with rewriting the left hand side of (5.1). Since $F\left(x, M^\theta(\rho^x)\right)$ is continuous in $x \in \Gamma$, the weak convergence result implies that

$$\int_\Gamma F\left(x, M^\theta(\rho^x)\right) m^\theta(dx) = \lim_{n\to\infty} \int_\Gamma F\left(x, M^\theta(\rho^x)\right) m^\theta_{\epsilon_n}(dx),$$

which, by the dominated convergence theorem, leads to

$$\begin{aligned}&\int_\Theta \int_\Gamma F\left(x, M^\theta(\rho^x)\right) m^\theta(dx)\, \mathcal{W}(d\theta) = \\ &\qquad \lim_{n\to\infty} \int_\Theta \int_\Gamma F\left(x, M^\theta(\rho^x)\right) E^\theta_{\epsilon_n}(x)\, dx \mathcal{W}(d\theta)\end{aligned}. \tag{5.2}$$

By Fubini's Theorem and the consideration (about viewing $E^\theta_{\epsilon_n}(x)$ as the Radon-Nikodym derivative of the translated Wiener measure) we made before the proof,

we have that the right hand side of (5.2) equals

$$\lim_{n\to\infty}\int_\Gamma\int_\Theta F\left(x, M^{\theta+\gamma h_{\mu^x_{\epsilon_n}}}\left(\rho^x\right)\right)\mathcal{W}\left(d\theta\right)dx.$$

Now given $x\in\mathbb{R}^4$, the Cameron-Martin theorem guarantees that also with probability 1, $m_{\epsilon_k}^{\theta+\gamma h_{\mu^x_{\epsilon_n}}}\left(dy\right)$ weakly converges to $m^{\theta+\gamma h_{\mu^x_{\epsilon_n}}}\left(dy\right)$ as $k\to\infty$ simultaneously for all $n\geq 1$. In particular,

$$\begin{aligned}M^{\theta+\gamma h_{\mu^x_{\epsilon_n}}}\left(\rho^x\right) &= \lim_{k\to\infty}M_{\epsilon_k}^{\theta+\gamma h_{\mu^x_{\epsilon_n}}}\left(\rho^x\right)\\ &= \lim_{k\to\infty}\int_{\mathbb{R}^4}\rho^x\left(y\right)\exp\left(\gamma^2\mathbb{E}\left[\mathcal{I}\left(h_{\mu^y_{\epsilon_k}}\right)\mathcal{I}\left(h_{\mu^x_{\epsilon_n}}\right)\right]\right)E^\theta_{\epsilon_k}\left(y\right)dy.\end{aligned}$$

At this point, it is clear that (5.1) would follow if we can show that for every $\theta\in\Theta_x$,

$$\begin{aligned}&\lim_{n\to\infty}\lim_{k\to\infty}\int_{\mathbb{R}^4}\rho^x\left(y\right)\exp\left(\gamma^2\mathbb{E}\left[\mathcal{I}\left(h_{\mu^y_{\epsilon_k}}\right)\mathcal{I}\left(h_{\mu^x_{\epsilon_n}}\right)\right]\right)E^\theta_{\epsilon_k}\left(y\right)dy\\ &\qquad=\int_{\mathbb{R}^4}\rho^x\left(y\right)\exp\left(\frac{\gamma^2}{2\pi^2}K_0\left(|x-y|\right)\right)m^\theta\left(dy\right)<\infty.\end{aligned}\tag{5.3}$$

The right hand side of (5.3) is finite for $\theta\in\Theta_x$ as we have seen in the proof of the first part of this lemma. To establish the equation in (5.3), we assume $n\geq 1$ is sufficiently large and $k\geq n$ and divide the integral in the left hand side of (5.3) into three pieces:

$$\begin{aligned}&\left\{\int_{|y-x|<\epsilon_n-\epsilon_k}+\int_{\epsilon_n-\epsilon_k\leq|y-x|\leq\epsilon_n+\epsilon_k}+\int_{|y-x|>\epsilon_n+\epsilon_k}\right\}\\ &\qquad\rho^x\left(y\right)\exp\left\{\gamma\mathcal{I}\left(h_{\mu^y_{\epsilon_k}}\right)\left(\theta\right)+\gamma^2\mathbb{E}\left[\mathcal{I}\left(h_{\mu^y_{\epsilon_k}}\right)\mathcal{I}\left(h_{\mu^x_{\epsilon_n}}\right)\right]-\frac{\gamma^2}{2}G\left(\epsilon_k\right)\right\}dy.\end{aligned}\tag{5.4}$$

We will investigate the limit as $k\to\infty$ and then $n\to\infty$ of each piece separately.

By (2.10), the last integral in (5.4) equals

$$\int_{|y-x|>\epsilon_n+\epsilon_k}\rho^x\left(y\right)e^{\frac{\gamma^2}{2\pi^2}K_0\left(|x-y|\right)}E^\theta_{\epsilon_k}\left(y\right)dy.\tag{5.5}$$

If the domain in (5.5) is replaced by $\{y:\,|y-x|>\epsilon_n\}$, then the integral would be

$$\begin{aligned}&\int_{|y-x|>\epsilon_n}\rho^x\left(y\right)e^{\frac{\gamma^2}{2\pi^2}K_0\left(|x-y|\right)}E^\theta_{\epsilon_k}\left(y\right)dy=\\ &\qquad M^\theta_{\epsilon_k}\left(\rho^x e^{\frac{\gamma^2}{2\pi^2}K_0\left(|x-\cdot|\vee\epsilon_n\right)}\right)-e^{\frac{\gamma^2}{2\pi^2}K_0\left(\epsilon_n\right)}\int_{|x-y|\leq\epsilon_n}\rho^x\left(y\right)E^\theta_{\epsilon_k}\left(y\right)dy.\end{aligned}\tag{5.6}$$

As $k\to\infty$ and then $n\to\infty$, the first term in the right hand side of (5.6) converges to $\int_{\mathbb{R}^4}\rho^x\left(y\right)e^{\frac{\gamma^2}{2\pi^2}K_0\left(|x-y|\right)}m^\theta\left(dy\right)$ (which is the term we want and also the only term that should survive in the limit). On the other hand, as $k\to\infty$, the second term on the right hand side of (5.6) is bounded by $e^{\frac{\gamma^2}{2\pi^2}K_0\left(\epsilon_n\right)}m^\theta\left(\overline{B_{\epsilon_n}\left(x\right)}\right)$, which, because $\theta\in\Theta_x$, converges to zero when $n\to\infty$. As for the "redundant annulus" which is the difference between the left hand side of (5.6) and (5.5), it's bounded

by $e^{\frac{\gamma^2}{2\pi^2}K_0(\epsilon_n)}$ times the integral of $E^\theta_{\epsilon_k}$ over the annulus $\{\epsilon_n < |x-\cdot| \leq \epsilon_n + \epsilon_k\}$. One can apply the Schwarz inequality to see that this integral is bounded by

$$e^{\frac{\gamma^2}{2}G(\epsilon_k)}\mathrm{vol}\left(\{\epsilon_n < |x-\cdot| \leq \epsilon_n + \epsilon_k\}\right)^{\frac{1}{2}}\left(\overline{m^{\theta,2\gamma}}\,(2\Gamma)\right)^{\frac{1}{2}} \tag{5.7}$$

where $2\Gamma \equiv \{2y : y \in \Gamma\}$. Given the considerations at the end of Section 2, without loss of generality, we can assume $\overline{m^{\theta,2\gamma}}\,(2\Gamma)$ is finite and hence the limit of (5.7) as $k \to \infty$ with n fixed is zero (because the volume of the annulus $\approx \epsilon_k \approx e^{-2\pi^2 G(\epsilon_k)}$ and $2\pi^2 > \gamma^2$), so this "annulus" is negligible.

The second integral in (5.4) becomes negligible by a similar argument. This time the Schwarz inequality implies that the second integral is bounded by

$$e^{\gamma^2\sqrt{G(\epsilon_k)G(\epsilon_n)}}e^{\frac{\gamma^2}{2}G(\epsilon_k)}\mathrm{vol}^{\frac{1}{2}}\left(\{\epsilon_n - \epsilon_k < |x-\cdot| < \epsilon_n + \epsilon_k\}\right)\left(\overline{m^{\theta,2\gamma}}\,(2\Gamma)\right)^{\frac{1}{2}}$$

where we use the simple estimate $\mathbb{E}\left[\left|\mathcal{I}\left(h_{\mu^y_{\epsilon_k}}\right)\mathcal{I}\left(h_{\mu^x_{\epsilon_n}}\right)\right|\right] \leq \sqrt{G(\epsilon_k)\,G(\epsilon_n)}$. Again, with n fixed, the factor that involves k converges to zero.

As for the first integral, because of (2.9) and the asymptotics of the Bessel functions involved, $\mathbb{E}\left[\mathcal{I}\left(h_{\mu^y_{\epsilon_k}}\right)\mathcal{I}\left(h_{\mu^x_{\epsilon_n}}\right)\right]$ is bounded by $\eta G(\epsilon_n)$ for some constant $\eta \in (1,2)$ for all sufficiently large n. Therefore, with n fixed, the integral as $k \to \infty$ is bounded by $e^{\eta\gamma^2 G(\epsilon_n)}m^\theta\left(\overline{B_{\epsilon_n}(x)}\right)$, and as $n \to \infty$ it therefore converges to zero.

So far we have proved the claim (5.1). To reach (4.5), one takes a sequence $\{\rho^x_l : l \geq 1\} \subseteq C^\infty_c\left(\mathbb{R}^4\right)$ such that $0 \leq \rho^x_l \nearrow \chi_{B_r(x)}$ as $l \to \infty$, and for every $l \geq 1$, $x \in \mathbb{R}^4 \mapsto \rho^x_l \in C_0\left(\mathbb{R}^4\right)$ is continuous. So (5.1) holds for each $l \geq 1$. After carefully examining the integrals on both sides of (5.1), one realizes that the limit as $l \to \infty$ can be passed all the way inside to produce (4.5). At the end, it's clear that given x, the distribution of $\hat{m}^{\theta,x}(B_r(x))$ under $\mathcal{W}$ is independent of x due to the translation invariance of measure $m^\theta(dy)$. Hence we have completed the proof to Lemma 6. □

Now we move on to the proofs of the KPZ results. The techniques we adopt here differ from those used in the two dimensional proofs in [DS11], partly because of the absence in our setting of the two dimensional conformal structure as well as the compactness of the domain. For example, the next lemma is the "tail estimate" which is the key estimate in proving both Lemma 8 and Theorem 9. In the two dimensional counterpart, the corresponding estimate ([DS11], §4.3) is a super-exponential type of estimate. Below we prove an exponential type of estimate in the four dimensional setting, but by carefully "tuning" the exponential decay rate, we can still make it sufficient for our purposes. Again, the occurrence of $\hat{m}^\theta(dy)$ in the next lemma refers to the measure $e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)}m^\theta(dy)$ assuming $\theta \in \Theta_0$. In other words, only balls centered at the origin are concerned. However, since the distribution of $\hat{m}^{\theta,x}(B_r(x))$ under $\mathcal{W}$ does not depend on x, the same result will hold for $\hat{m}^{\theta,x}(dy) = e^{\frac{\gamma^2}{2\pi^2}K_0(|x-y|)}m^\theta(dy)$ (assuming $\theta \in \Theta_x$) no matter what x is.

Lemma 10. *Let B be the closed ball in $\mathbb{R}^4$ centered at the origin with unit volume under $e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)}dy$, i.e., $\int_B e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)}dy = 1$. If δ and ρ are constants satisfying*

$$0 < \delta < 4\pi^2 - 2\gamma^2 \text{ and } \frac{4\pi^2+\gamma^2}{8\pi^2-\gamma^2-\delta} < \rho < 1,$$

then there exists $C>0$ such that[9] *for all sufficiently large $A>0$,*

$$\mathcal{W}\left(\hat{m}^{\theta}(B)\leq e^{-A\gamma}\right)\leq C\exp\left[-\frac{2\rho}{\gamma}\left(8\pi^2-\gamma^2-\frac{\gamma^2}{\rho}-\delta\right)A\right]. \tag{5.8}$$

Proof. Since B is closed, it suffices to estimate $\mathcal{W}\left(\limsup_{n\to\infty}\hat{m}^{\theta}_{\epsilon_n}(B)\leq e^{-A\gamma}\right)$ where $\hat{m}^{\theta}_{\epsilon_n}(dy)$ has density $e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)}$ with respect to $m^{\theta}_{\epsilon_n}(dy)$. By the same argument as used in deriving the estimate (3.1), we can show that there exists constant $C>0$ such that for all $n\geq 1$,

$$\mathbb{E}^{\mathcal{W}}\left[\left|\hat{m}^{\theta}_{\epsilon_{n+1}}(B)-\hat{m}^{\theta}_{\epsilon_n}(B)\right|^2\right]\leq Ce^{-\left(8\pi^2-\gamma^2\right)G(\epsilon_n)}.$$

For any δ with $0<\delta<4\pi^2-2\gamma^2$, denote $\mathcal{A}'_n$, $n\geq 1$, the measurable set

$$\left\{\forall l\geq n,\left|\hat{m}^{\theta}_{\epsilon_{l+1}}(B)-\hat{m}^{\theta}_{\epsilon_l}(B)\right|\leq e^{-A\gamma}e^{-\frac{\delta}{2}G(\epsilon_l)}\right\}.$$

Then it follows easily from Chebyshev's inequality and the Borel-Cantelli Lemma that $\mathcal{W}\left(\bigcup_{n=1}^{\infty}\mathcal{A}'_n\right)=1$. Moreover, if $\mathcal{A}_1=\mathcal{A}'_1$ and $\mathcal{A}_n=\mathcal{A}'_n\backslash\mathcal{A}'_{n-1}$ for $n\geq 2$, then there exists constant $C>0$ such that for all $n\geq 2$,

$$\mathcal{W}\left(\mathcal{A}_n\right)\leq Ce^{2A\gamma}e^{-\left(8\pi^2-\gamma^2-\delta\right)G(\epsilon_n)}. \tag{5.9}$$

Set $\mathcal{B}\equiv\left\{\limsup_{n\to\infty}\hat{m}^{\theta}_{\epsilon_n}(B)\leq e^{-A\gamma}\right\}$, then $\mathcal{W}(\mathcal{B})=\sum_{n=1}^{\infty}\mathcal{W}(\mathcal{B}\cap\mathcal{A}_n)$ and it's clear that $\theta\in\mathcal{B}\cap\mathcal{A}_n$ implies $\hat{m}^{\theta}_{\epsilon_n}(B)\leq c_\delta e^{-A\gamma}$ where $c_\delta=1+\sum_{n=1}^{\infty}e^{-\frac{\delta}{2}G(\epsilon_n)}$. Given any ρ such that $\frac{4\pi^2+\gamma^2}{8\pi^2-\gamma^2-\delta}<\rho<1$ (notice that such ρ always exists since $0<\delta<4\pi^2-2\gamma^2$ and $4\pi^2+\gamma^2<8\pi^2-\gamma^2-\delta$), we set up the "threshold" $N\in\mathbb{N}$ which is the unique (recall that G is strictly decreasing on $(0,\infty)$) integer such that

$$G(\epsilon_N)<\frac{2\rho A}{\gamma}\text{ but }G(\epsilon_{N+1})\geq\frac{2\rho A}{\gamma}. \tag{5.10}$$

The desired estimate (5.8) is trivial when $n\geq N+1$, because (5.9) and (5.10) implies

$$\begin{aligned}\sum_{n=N+1}^{\infty}\mathcal{W}(\mathcal{B}\cap\mathcal{A}_n) &\leq Ce^{2A\gamma}e^{-\left(8\pi^2-\gamma^2-\delta\right)G(\epsilon_{N+1})}\\ &\leq C\exp\left[-\frac{2\rho}{\gamma}\left(8\pi^2-\gamma^2-\frac{\gamma^2}{\rho}-\delta\right)A\right].\end{aligned}$$

When $n=1,\cdots,N$, we apply Jensen's inequality to see that

$$\begin{aligned}&\mathcal{W}\left(\hat{m}^{\theta}_{\epsilon_n}(B)\leq c_\delta e^{-A\gamma}\right)\\ &\leq\mathcal{W}\left(\exp\left[\int_B\left(\gamma\mathcal{I}\left(h_{\mu^y_{\epsilon_n}}\right)(\theta)-\frac{\gamma^2}{2}G(\epsilon_n)\right)e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)}dy\right]\leq c_\delta e^{-A\gamma}\right)\\ &\leq\mathcal{W}\left(\int_B\mathcal{I}\left(h_{\mu^y_{\epsilon_n}}\right)(\theta)\,e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)}dy\leq -A+\frac{\gamma}{2}G(\epsilon_n)+\frac{\log c_\delta}{\gamma}\right).\end{aligned} \tag{5.11}$$

By Corollary 3, without loss of generality, we can assume that for all $n\geq 1$ and every θ, the function $y\in B\mapsto\mathcal{I}\left(h_{\mu^y_{\epsilon_n}}\right)(\theta)$ is continuous and hence uniformly

[9] Throughout this section, C denotes a constant that may depend on γ, δ, ρ and R, but universal in A, ϵ_n, x and Λ. The values of C may change from line to line.

continuous on B. Therefore, one can easily check (for example, by writing the integral as the limit of a discrete sum of Gaussian random variables) that

$$\theta \in \Theta \mapsto \int_B \mathcal{I}\left(h_{\mu^y_{\epsilon_n}}\right)(\theta)\, e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)}dy \in \mathbb{R}$$

is also a centered Gaussian random variable for every $n \geq 1$, and furthermore, the variance can be bounded by a constant M that is universal in $n \geq 1$. In fact, M can be taken as a constant multiple of

$$\iint_{B\times B} K_0\left(|x-y|\right) e^{\frac{\gamma^2}{2\pi^2}(K_0(|x|)+K_0(|y|))}dxdy.$$

Since $G\left(\epsilon_n\right) < \frac{2\rho A}{\gamma}$ for $n = 1, \cdots, N$, $A - \frac{\gamma}{2}G\left(\epsilon_n\right) > (1-\rho)A$, and (5.11) implies that when A is sufficiently large,

$$\begin{aligned}
\mathcal{W}\left(\hat{m}^\theta_{\epsilon_n}(B) \leq c_\delta e^{-A\gamma}\right) &\leq \exp\left[-\frac{1}{2M}\left(A - \frac{\gamma}{2}G\left(\epsilon_n\right) - \frac{1}{\gamma}\log c_\delta\right)^2\right] \\
&\leq \exp\left[-\frac{1}{2M}\left((1-\rho)\,A - \frac{1}{\gamma}\log c_\delta\right)^2\right] \\
&\leq \exp\left[-\frac{1}{4M}(1-\rho)^2 A^2\right].
\end{aligned}$$

In addition, (5.10) implies that N is approximately a constant multiple of A. Therefore, when A is large,

$$\sum_{n=1}^N \mathcal{W}\left(\mathcal{B}\cap\mathcal{A}_n\right) \leq \sum_{n=1}^N \mathcal{W}\left(\hat{m}^\theta_{\epsilon_n}(B) \leq c_\delta e^{-A\gamma}\right) \leq CAe^{-\frac{(1-\rho)^2A^2}{4M}}.$$

So $\sum_{n=1}^N \mathcal{W}\left(\mathcal{B}\cap\mathcal{A}_n\right)$ actually decays super-exponentially fast as $A \to \infty$, and this estimate can certainly be transformed into the desired form as in (5.8). □

We are now ready to prove Lemma 8. Recall the notation $\hat{r}_\Lambda : \theta \mapsto \hat{r}_\Lambda(0,\theta)$ where $\hat{r}_\Lambda(0,\theta)$ is as defined in (4.6) with x being the origin. Let $(\kappa, K) \in [0,1]^2$ be a pair as in (4.14). In order to get (4.15), it suffices to show that

$$C^{-1} \leq \Lambda^{-K}\mathbb{E}^{\mathcal{W}}\left[\left(\hat{r}_\Lambda\right)^{4\kappa}\right] \leq C \tag{5.12}$$

for some constant $C > 0$ universal in Λ as $\Lambda \downarrow 0$. We will prove the existence of the upper bound and the lower bound in (5.12) separately.

Proof of the upper bound in (5.12): For notational convenience, we introduce the "stopping time" corresponding to $\hat{r}_\Lambda$, i.e., $T_\Lambda \equiv G\left(\hat{r}_\Lambda\right) - G\left(R\right)$. We want to show $\Lambda^{-K}\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T_\Lambda\right)\right]$ is bounded from above uniformly in small Λ. It's clear, from (5.8) and the fact that $\mathcal{W}\left(\Theta_0\right) = 1$, where Θ_0 is as in (3.4), that $T_\Lambda \in \left(-G\left(R\right), \infty\right)$ almost surely. Let constant δ and ρ be as in the statement of Lemma 10. Set

$$S \equiv \frac{-\log\Lambda}{8\pi^2-\gamma^2}\frac{2\rho\left(8\pi^2-\gamma^2-\frac{\gamma^2}{\rho}-\delta\right)-K\gamma^2}{2\rho\left(8\pi^2-\gamma^2-\frac{\gamma^2}{\rho}-\delta\right)}.$$

Then the expectation of $\exp\left(-8\pi^2\kappa T_\Lambda\right)$ can be written as

$$\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T_\Lambda\right)\chi_{\{-G(R)<T_\Lambda<S\}}\right]+\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T_\Lambda\right)\chi_{\{S\le T_\Lambda<\infty\}}\right].$$

In the first term, $T_\Lambda<S$ implies that the volume of the closed ball centered at the origin with radius $r(S)=G^{-1}(S+G(R))$ ($\approx\exp\left(-2\pi^2 S\right)$) is no greater than Λ under the measure $\hat{m}^\theta(dy)$, while this ball has volume $\approx(r(S))^{4-\frac{\gamma^2}{2\pi^2}}$ ($\approx\exp\left(-\left(8\pi^2-\gamma^2\right)S\right)$) under the measure $e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)}dy$. However, by Lemma 10, the probability of this event is bounded by

$$C\exp\left(-\frac{2\rho}{\gamma}\left(8\pi^2-\gamma^2-\frac{\gamma^2}{\rho}-\delta\right)\left(\frac{-\log\Lambda}{\gamma}-\frac{8\pi^2-\gamma^2}{\gamma}S\right)\right),$$

which, given this particular choice of S, is equal to a constant multiple of Λ^K. Therefore, the first piece of integral causes no trouble.

The second integral is bounded by $e^{-8\pi^2\kappa S}$. Hence we only need to check that $\Lambda^{-K}e^{-8\pi^2\kappa S}$, or equivalently, $\exp\left(-K\log\Lambda-8\pi^2\kappa S\right)$ stays bounded as $\Lambda\downarrow 0$. In fact, we will show that for all possible values of (κ,K) and all sufficiently small $\Lambda>0$, $K\log\Lambda+8\pi^2\kappa S\ge 0$, that is (assuming $\log\Lambda<0$),

$$K\le\frac{8\pi^2\kappa}{8\pi^2-\gamma^2}\frac{2\rho\left(8\pi^2-\gamma^2-\frac{\gamma^2}{\rho}-\delta\right)-K\gamma^2}{2\rho\left(8\pi^2-\gamma^2-\frac{\gamma^2}{\rho}-\delta\right)}. \tag{5.13}$$

To simplify the notations, let's write $\zeta\equiv 2\rho\left(8\pi^2-\gamma^2-\frac{\gamma^2}{\rho}-\delta\right)$. Recall from the statement of Lemma 10 that $0<\delta<4\pi^2-2\gamma^2$ and $\frac{4\pi^2+\gamma^2}{8\pi^2-\gamma^2-\delta}<\rho<1$, so $\zeta>8\pi^2$. If we express κ in terms of K according to (4.14), then the statement in (5.13) is equivalent to

$$F(K)\equiv\gamma^2K^2+\left(16\pi^2-\gamma^2-\zeta\right)K-\zeta\le 0$$

for all possible values of $K\in[0,1]$. However, this is clearly true since F is quadratic and $F(0)=-\zeta<0$ as well as $F(1)=16\pi^2-2\zeta<0$. □

Proof of the lower bound in (5.12): Recall that T^*_Λ is the stopping time (as defined in (4.10)) associated with the "approximating" measure $\hat{m}^{\theta*}$, and

$$\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T^*_\Lambda\right)\right]=\Lambda^K.$$

We observe that $\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T_\Lambda\right)\right]$ is greater than the integral of $\exp\left(-8\pi^2\kappa T_\Lambda\right)$ over the subset $\{T_\Lambda\le T^*_\Lambda\}$, where the integrand is greater or equal to $\exp\left(-8\pi^2\kappa T^*_\Lambda\right)$. Therefore, we have

$$\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T_\Lambda\right)\right]\ge\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T^*_\Lambda\right)\right]-\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T^*_\Lambda\right)\chi_{\left\{T_\Lambda>T^*_\Lambda\right\}}\right].$$

It is clear that in order to get the desired lower bound, we need to find constant $0<c<1$ such that

$$\Lambda^{-K}\mathbb{E}^{\mathcal{W}}\left[\exp\left(-8\pi^2\kappa T^*_\Lambda\right)\chi_{\left\{T_\Lambda>T^*_\Lambda\right\}}\right]\le c \tag{5.14}$$

uniformly in small Λ. Conditioning on $T^*_\Lambda=T$, $T_\Lambda>T$ implies $\hat{m}^\theta\left(B_{r(T)}\right)>\Lambda$ and hence $\frac{\hat{m}^\theta\left(B_{r(T)}\right)}{\hat{m}^{\theta*}\left(B_{r(T)}\right)}>1$. By Chebyshev's inequality, other than a factor given by the probability density function of T^*_Λ, the conditional probability of $\{T_\Lambda>T\}$

is bounded by the expectation of $\frac{\hat{m}^{\theta}\left(B_{r(T)}\right)}{\hat{m}^{\theta *}\left(B_{r(T)}\right)}$, which, given the expression in (4.8) (which is the conditional expectation of the numerator given the denominator), can be bounded by constant $c \in (0,1)$ which is universal in Λ and T. So the estimate in (5.14) will be satisfied by this choice of c. □

Proof of Theorem 9: Assume $D \subseteq \overline{B_N(0)}$ for some sufficently large $N \geq 1$. Let $r_\Lambda(x,\theta)$ and $D^{\Lambda,\theta}$ be as defined in (4.1) and (4.2). Denote $\mathcal{N}(d\theta dx) \equiv \mathcal{W}(d\theta)\,dx$. Based on Lemma 6, $\mathbb{E}^{\mathcal{W}}\left[m^{\theta}\left(D^{\Lambda,\theta}\right)\right]$ equals

$$\begin{aligned}
& \mathcal{M}\left(\{(x,\theta):\ \text{either } x \in D \text{ or } \operatorname{dist}(x,D) < r_\Lambda(x,\theta)\}\right) \\
(5.15)\quad = \ & \mathcal{N}\left(\{(x,\theta): |x| \leq 2N, \operatorname{dist}(x,D) < \hat{r}_\Lambda(x,\theta)\}\right) \\
& + \lim_{2N \leq M \to \infty} \mathcal{N}\left(\{(x,\theta): 2N \leq |x| \leq M,\ \operatorname{dist}(x,D) < \hat{r}_\Lambda(x,\theta)\}\right).
\end{aligned}$$

In the first term of the right hand side of (5.15), conditioning on $\hat{r}_\Lambda(x,\theta)$ under $\mathcal{N}(d\theta dx)$, since its marginal distribution on Θ does not depend on x, the conditional probability of the set is proportional to $\operatorname{vol}\left(D_{\hat{r}_\Lambda(x,\theta)} \cap \overline{B_{2N}(0)}\right)$. We further split the set into two cases: $\hat{r}_\Lambda(x,\theta) > N$ and $\hat{r}_\Lambda(x,\theta) \leq N$, the later of which also implies $D_{\hat{r}_\Lambda(x,\theta)} \subseteq \overline{B_{2N}(0)}$. Therefore, the first term can be rewritten as (up to a constant depending on N)

$$\begin{aligned}
& \mathbb{E}^{\mathcal{W}}\left[\operatorname{vol}\left(D_{\hat{r}_\Lambda(x,\theta)}\right)\chi_{\{\hat{r}_\Lambda(x,\theta)\leq N\}}\right] + \mathbb{E}^{\mathcal{W}}\left[\operatorname{vol}\left(D_{\hat{r}_\Lambda(x,\theta)} \cap \overline{B_{2N}(0)}\right)\chi_{\{\hat{r}_\Lambda(x,\theta)>N\}}\right] \\
= & \mathbb{E}^{\mathcal{W}}\left[\operatorname{vol}\left(D_{\hat{r}_\Lambda(x,\theta)}\right)\right] - \mathbb{E}^{\mathcal{W}}\left[\operatorname{vol}\left(D_{\hat{r}_\Lambda(x,\theta)}\right)\chi_{\{\hat{r}_\Lambda(x,\theta)>N\}}\right] \\
& \qquad + \mathbb{E}^{\mathcal{W}}\left[\operatorname{vol}\left(D_{\hat{r}_\Lambda(x,\theta)} \cap \overline{B_{2N}(0)}\right)\chi_{\{\hat{r}_\Lambda(x,\theta)>N\}}\right].
\end{aligned}$$

According to the assumption (4.12) and Lemma 8, $\mathbb{E}^{\mathcal{W}}\left[\operatorname{vol}\left(D_{\hat{r}_\Lambda(x,\theta)}\right)\right] \approx \Lambda^K$ when Λ is sufficiently small. On the other hand, given $\hat{r}_\Lambda(x,\theta) > N$, $D_{\hat{r}_\Lambda(x,\theta)}$ is always contained in the ball centered at the origin with radius $2\hat{r}_\Lambda(x,\theta)$, so the last two terms in the right hand side of the equation above are both bounded by (up to a constant)

$$\mathbb{E}^{\mathcal{W}}\left[\left(\hat{r}_\Lambda(x,\theta)\right)^4 \chi_{\{\hat{r}_\Lambda(x,\theta)>N\}}\right] \leq 4\int_{[1,\infty)} u^3 \mathcal{W}\left(\hat{r}_\Lambda(x,\theta) > u\right) du. \tag{5.16}$$

If $\zeta \equiv 2\rho\left(8\pi^2 - \gamma^2 - \frac{\gamma^2}{\rho} - \delta\right)$ where δ and ρ are the same as in the statement of Lemma 10, then by (5.8),

$$\mathcal{W}\left(\hat{r}_\Lambda(x,\theta) > u\right) \leq \mathcal{W}\left(\hat{m}^{\theta,x}\left(B_u(x)\right) \leq \Lambda\right) \leq C\Lambda^{\frac{\zeta}{\gamma^2}} u^{-\frac{\zeta}{\gamma^2}\left(4 - \frac{\gamma^2}{2\pi^2}\right)}.$$

Given the particular range of δ, ρ and ζ, one sees that not only is the integral in (5.16) finite, but it also converges to zero faster than Λ^K as $\Lambda \downarrow 0$ for any possible value of $K \in [0,1]$.

In the second term in (5.15), since $D \subseteq \overline{B_N(0)}$, the assumptions $|x| \geq 2N$ and $\operatorname{dist}(x,D) < \hat{r}_\Lambda(x,\theta)$ imply $\hat{r}_\Lambda(x,\theta) > \frac{1}{2}|x|$ whose probability, as we have seen earlier, is bounded by $C\Lambda^{\frac{\zeta}{\gamma^2}}|x|^{-\frac{\zeta}{\gamma^2}\left(4-\frac{\gamma^2}{2\pi^2}\right)}$ which is integrable (with respect to dx) in the entire domain $\{|x| \geq 2N\}$. Therefore the second term also converges to zero faster than Λ^K as $\Lambda \downarrow 0$. To summarize, we have shown that $\mathbb{E}^{\mathcal{W}}\left[m^{\theta}\left(D^{\Lambda,\theta}\right)\right]$ is a constant multiple of $\Lambda^K + o\left(\Lambda^K\right)$ as $\Lambda \downarrow 0$ which is sufficient for the desired conclusion. □

6. Possible Generalizations

Generalizations to $\mathbb{R}^{2n}$: In this subsection, we outline a possible generalization of the four dimensional treatments carried out in previous sections to higher even dimensions $\mathbb{R}^{2n}$ with $n \geq 2$. We consider the GFF on $\mathbb{R}^{2n}$ with the underlying Hilbert space $H \equiv H^n\left(\mathbb{R}^{2n}\right)$ which is the completion of the Schwartz test function space $\mathcal{S}\left(\mathbb{R}^{2n}\right)$ under the inner product $\left((I-\Delta)^n \cdot, \cdot\right)_{L^2}$. Similarly, for every $x \in \mathbb{R}^{2n}$ and $\epsilon > 0$, σ_ϵ^x denotes the tempered distribution which is to take the spherical average of a test function over the sphere $S_\epsilon(x)$. In this setting, $\sigma_\epsilon^x \in H^{-n}\left(\mathbb{R}^{2n}\right)$ and again if $h_{\sigma_\epsilon^x} \equiv (I-\Delta)^{-n}\sigma_\epsilon^x$, then $h_{\sigma_\epsilon^x} \in H$ and the Paley-Wiener integral $\mathcal{I}\left(h_{\sigma_\epsilon^x}\right)$ can be viewed as the "generalized" action of σ_ϵ^x on the GFF. Moreover, the higher order of the operator $(I-\Delta)^n$ allows us to take higher "derivatives" of σ_ϵ^x in the radial variable ϵ. If $d^m\sigma_\epsilon^x \equiv \frac{d^m}{d\epsilon^m}\sigma_\epsilon^x$ is defined in the sense of tempered distribution for every $m \in \mathbb{N}$, then simple computations of the Fourier transforms show that

$$\hat{\sigma_\epsilon^x}(\xi) = C_n e^{i(x,\xi)_{\mathbb{R}^{2n}}} (\epsilon|\xi|)^{1-n} J_{n-1}(\epsilon|\xi|)$$

$$\text{and } \left(d^m\sigma_\epsilon^x\right)^{\hat{}}(\xi) = \frac{d^m}{d\epsilon^m}\hat{\sigma_\epsilon^x}(\xi) = C_n e^{i(x,\xi)_{\mathbb{R}^{2n}}} \frac{d^m}{d\epsilon^m}\left(\frac{J_{n-1}(\epsilon|\xi|)}{(\epsilon|\xi|)^{n-1}}\right),$$

where $C_n > 0$ is a dimensional constant. In particular, we can write

$$\left(d^m\sigma_\epsilon^x\right)^{\hat{}}(\xi) = C_n e^{i(x,\xi)_{\mathbb{R}^{2n}}} \varphi^{(m)}(\epsilon|\xi|)\,|\xi|^m$$

where ([Wat], §3.31) $\varphi(r) \equiv \frac{J_{n-1}(r)}{r^{n-1}}$ for $r > 0$ and $\varphi^{(m)}(r)$ is analytic in r near 0 and asymptotic to $r^{-\left(n-\frac{1}{2}\right)}$ as $r \to \infty$ for every $m \in \mathbb{N}$. Therefore, σ_ϵ^x and $d^m\sigma_\epsilon^x$ for $1 \leq m \leq n-1$ are in $H^{-n}\left(\mathbb{R}^{2n}\right)$.

We can mimic the approach in Section 2 and define the vector-valued Gaussian random variable on Θ: for every $x \in \mathbb{R}^{2n}$ and $\epsilon > 0$,

$$V_\epsilon^x \equiv \left(\mathcal{I}\left(h_{\sigma_\epsilon^x}\right), \mathcal{I}\left(h_{d\sigma_\epsilon^x}\right), \cdots, \mathcal{I}\left(h_{d^{n-1}\sigma_\epsilon^x}\right)\right)^\top.$$

It turns out that in this setting we can also compute the covariance matrix of the family $\left\{V_\epsilon^x : x \in \mathbb{R}^{2n}, \epsilon > 0\right\}$ explicitly under each circumstance as prescribed in Lemma 1, and the covariance matrix also has a similar "separability" property as in four dimensional case. In fact, following a similar computation as the one (provided in the appendix) conducted to prove Lemma 1, it is not hard to see that there exist invertible $n \times n$ matrices $\mathbf{A}(r)$, $\mathbf{B}(r)$, $\mathbf{C}(r)$ and $\mathbf{D}(r)$ for every $r \in (0,\infty)$, such that all the entries of $\mathbf{A}(r)$ and $\mathbf{D}(r)$ are functions in the linear span of $\left\{r^{-\ell}K_k(r) : 0 \leq \ell \leq k \leq 2n-2\right\}$, while all the entries of $\mathbf{B}(r)$ and $\mathbf{C}(r)$ are in the linear span of $\left\{r^{-\ell}I_k(r) : 0 \leq \ell \leq k \leq 2n-2\right\}$. Moreover, given $x \in \mathbb{R}^{2n}$ and $\epsilon_1 \geq \epsilon_2 > 0$, $\mathbb{E}^{\mathcal{W}}\left[V_{\epsilon_1}^x\left(V_{\epsilon_2}^x\right)^\top\right] = \mathbf{A}(\epsilon_1)\,\mathbf{B}^\top(\epsilon_2)$; given $x, y \in \mathbb{R}^{2n}$ with $x \neq y$ and $\epsilon_1 > |x-y| + \epsilon_2$, $\mathbb{E}^{\mathcal{W}}\left[V_{\epsilon_1}^x\left(V_{\epsilon_2}^y\right)^\top\right] = \mathbf{A}(\epsilon_1)\,\mathbf{C}(|x-y|)\,\mathbf{B}^\top(\epsilon_2)$; given $x, y \in \mathbb{R}^{2n}$ with $x \neq y$ and $|x-y| > \epsilon_1 + \epsilon_2$, $\mathbb{E}^{\mathcal{W}}\left[V_{\epsilon_1}^x\left(V_{\epsilon_2}^y\right)^\top\right] = \mathbf{B}(\epsilon_1)\,\mathbf{D}(|x-y|)\,\mathbf{B}^\top(\epsilon_2)$. Therefore, if we similarly defined the "normalized" vector $U_\epsilon^x \equiv \mathbf{B}^{-1}(\epsilon)\,V_\epsilon^x$, then the Gaussian family $\left\{U_\epsilon^x : x \in \mathbb{R}^{2n}, \epsilon > 0\right\}$ will have the same properties as those of the corresponding family (also denoted by U_ϵ^x) in four dimensions.

On the other hand, all the entries of the matrix $\mathbf{B}(\epsilon)$ are linear combinations of $\epsilon^{-l}I_k(\epsilon)$ with $0 \leq l \leq k \leq 2n-2$, and if one lets $\epsilon_2 \downarrow 0$ in the covariance matrix

obtained in the second circumstance (when $\epsilon_1 > |x-y| + \epsilon_2$) from above, combined with integral expressions for the entries of the covariance matrix, then one can easily conclude that there exists constant matrix $\mathbf{B}$ which is non-degenerate (hence so is $\mathbf{B}^{-1}$) such that $\mathbf{B}(\epsilon)$ converges to $\mathbf{B}$ as $\epsilon \downarrow 0$. Therefore, all the entries of $\mathbf{B}^{-1}(\epsilon)$ must be analytic in ϵ near zero. In particular, by examining the asymptotics of the entries of $\mathbf{B}^{-1}(\epsilon)$ near zero, one can find the appropriate constant vector $\zeta \in \mathbb{R}^{2n}$ such that $(U_\epsilon^x, \zeta)_{\mathbb{R}^{2n}}$ "approximates" the GFF at x when ϵ is small in the same sense as described in Section 2.

Clearly, $(U_\epsilon^x, \zeta)_{\mathbb{R}^{2n}}$ has all the properties of $\mathcal{I}\left(h_{\mu_\epsilon^x}\right)$ in four dimensions as stated in Theorem 2. When $\epsilon_1 \geq \epsilon_2 > 0$,

$$G(\epsilon_1) \equiv \mathbb{E}^{\mathcal{W}}\left[\left(U_{\epsilon_1}^x, \zeta\right)_{\mathbb{R}^{2n}}^2\right] = \mathbb{E}^{\mathcal{W}}\left[\left(U_{\epsilon_1}^x, \zeta\right)_{\mathbb{R}^{2n}}\left(U_{\epsilon_2}^x, \zeta\right)_{\mathbb{R}^{2n}}\right];$$

when $\epsilon_1 > \epsilon_2 + |x-y|$, $\mathbb{E}^{\mathcal{W}}\left[\left(U_{\epsilon_1}^x, \zeta\right)_{\mathbb{R}^{2n}}\left(U_{\epsilon_2}^y, \zeta\right)_{\mathbb{R}^{2n}}\right] = c(\epsilon_1, |x-y|)$ which is independent of ϵ_2; when $|x-y| > \epsilon_1 + \epsilon_2$, $\mathbb{E}^{\mathcal{W}}\left[\left(U_{\epsilon_1}^x, \zeta\right)_{\mathbb{R}^{2n}}\left(U_{\epsilon_2}^y, \zeta\right)_{\mathbb{R}^{2n}}\right] = d(|x-y|)$ which is independent of ϵ_1 and ϵ_2. In principle, we can derive the explicit formulas of $G(\epsilon_1)$, $c(\epsilon_1, |x-y|)$ and $d(|x-y|)$, and one can expect that they have logarithmic growth when ϵ_1 and $|x-y|$ are small because the Green's function of the operator $(I-\Delta)^n$ on $\mathbb{R}^{2n}$ has logarithmic growth near the diagonal. Therefore, it's reasonable to believe that if one takes $(U_\epsilon^x, \zeta)_{\mathbb{R}^{2n}}$ to construct a sequence of approximating measures, i.e.,

$$m_{\epsilon_k}^\theta(dx) \equiv \exp\left(\gamma\left(U_{\epsilon_k}^x, \zeta\right)_{\mathbb{R}^{2n}}(\theta) - \frac{\gamma^2}{2}G(\epsilon_k)\right)dx,$$

then the sequence $\left\{m_{\epsilon_k}^\theta : k \geq 1\right\}$ will almost surely admit a limit measure in the sense of weak convergence. Furthermore, the quantum scaling component of a bounded set on $\mathbb{R}^{2n}$ under this limit measure should also satisfy a quadratic relation with its counterpart under the Lebesgue measure. However, the amount and the complexity of computations quickly become considerable as n increases.

Generalizations to Manifolds: In this last part we explain a more conceptual approach to constructing analogues of the two-dimensional GFF on compact even-dimensional manifolds. As we have remarked in the introduction, in dimension two, the GFF defines a measure on a conformal class of metrics on a Riemann surface Σ, constructed starting with a reference metric g_0 on Σ, but in the end independent of g_0. In fact, the GFF inner product of two functions $f_1, f_2 \in C_c^\infty(\Sigma)$ is defined by ([Sh, DS11, DS09, HMP])

$$(f_1, f_2)_{\Delta_{g_0}} \equiv (f_1, \Delta_{g_0} f_2) \equiv \int_\Sigma f_1(x)(\Delta_{g_0} f_2)(x) d\mathrm{vol}_{g_0}(x),$$

where Δ_{g_0} is the Laplace-Beltrami operator on Σ with respect to g_0. This inner product is conformally invariant. Indeed, if the metric g_0 is changed conformally to $g_1 = e^{2\omega} g_0$ for some $\omega \in C_c^\infty(\Sigma)$, then the volume element changes as

$$d\mathrm{vol}_{g_1} = e^{2\omega} d\mathrm{vol}_{g_0},$$

while the Laplacian is changed as $\Delta_{g_1} = e^{-2\omega}\Delta_{g_0}$. Therefore, after obvious cancellations we find that

$$(f_1, f_2)_{\Delta_{g_1}} = (f_1, f_2)_{\Delta_{g_0}}.$$

It seems natural to define a similar measure for conformal classes of metrics in higher dimensions. Below, we explain how to do that for certain conformal

classes on compact manifolds $\mathfrak{M}$ of even dimension $2n$. In the construction, we find it convenient to use *conformally covariant* elliptic operators described below. In the discussion, we restrict ourselves to even-dimensional manifolds, although the corresponding operators can be defined in odd dimensions as well.

Let $\mathfrak{M}$ be a manifold of even dimension $2n$, $n \geq 2$, and g_0 a Riemannian metric on $\mathfrak{M}$. Then, there exists on $\mathfrak{M}$ an elliptic operator $P = P_{g_0}$ of order $2n$, called the *dimension-critical GJMS operator*, constructed by Graham-Jenne-Mason-Sparling in [GJMS], with the following properties:

$$P = \Delta^n + \text{ lower order terms;}$$

in fact, P has a polynomial expression in (Levi-Civita connection) ∇ and (scalar curvature) R, with coefficients that are rational in dimension $2n$; P is formally self-adjoint ([GZ, FG]); under a conformal change of metric $g_1 = e^{2\omega} g_0$, the operator P changes as $P_{g_1} = e^{-2n\omega} P_{g_0}$.

Given these properties, we can imitate the construction of the GFF in dimension 2: for $f_1, f_2 \in C^\infty(\mathfrak{M})$, the inner product is defined by

$$(f_1, f_2)_{P_{g_0}} \equiv \int_{\mathfrak{M}} f_1(x)(P_{g_0} f_2)(x) d\mathrm{vol}_{g_0}(x).$$

Then, this inner product is also conformally invariant. When the metric g_0 is changed conformally to $g_1 = e^{2\omega} g_0$, the volume element changes as

$$d\mathrm{vol}_{g_1} = e^{2n\omega} d\mathrm{vol}_{g_0},$$

while P changes as $P_{g_1} = e^{-2n\omega} P_{g_0}$. Again, we get the relation

$$(f_1, f_2)_{P_{g_1}} = (f_1, f_2)_{P_{g_0}},$$

just like in dimension two.

When $n = 2$, the dimension-critical GJMS operator

$$P_4 = \Delta_{g_0}^2 + \delta[(2/3) R_{g_0} g_0 - 2\mathrm{Ric}_{g_0}] d$$

is also called the *Paneitz operator*. If $\mathfrak{M}$ is flat, then the Paneitz operator is equal to Δ^2, hence in $\mathbb{R}^4$ it is natural to work with Δ^2. However, since $\mathbb{R}^4$ is not compact, we need to consider the operator on a compact domain, in which case we have to choose proper boundary operators in order to preserve the conformal covariance property. This will be further explored in future work.

On the compact $2n$-dimensional manifold $\mathfrak{M}$, if we construct a Gaussian random field using the dimension-critical GJMS operator P, then the covariance function of the field is given by the Green's function $G_P(x, y)$ of the operator P. Let $d(x, y)$ be the Riemannian distance between x and y on $\mathfrak{M}$. Then, it is known ([CY, Nd, Pon]) that as $d(x, y) \downarrow 0$, $G_P(x, y)$ is asymptotic to $-C_n \log d(x, y)$ where $C_n > 0$ depends only on the dimension. This is similar to the well-known behavior of the Green's function of the Laplace-Beltrami operator Δ in dimension two. This will become an important ingredient in the construction of the random measure on the manifold, which we intend to explore in a future paper.

7. Appendix

This section contains all the computations with the Bessel functions. We start with the Fourier transforms of σ_ϵ^x and $d\sigma_\epsilon^x$, and list all the integral expressions for the covariance function of the family $\left\{\mathcal{I}\left(h_{\sigma_\epsilon^x}\right), \mathcal{I}\left(h_{d\sigma_\epsilon^x}\right) : x \in \mathbb{R}^4, \epsilon > 0\right\}$.

Lemma 11. *Recall from (2.1) and (2.2) that the Fourier transforms of σ_ϵ^x and $d\sigma_\epsilon^x$ are, respectively,*

$$\hat{\sigma_\epsilon^x}(\xi) = 2\left(\epsilon\,|\xi|\right)^{-1} J_1\left(\epsilon\,|\xi|\right) e^{i(x,\xi)_{\mathbb{R}^4}}$$

$$\text{and } \hat{d\sigma_\epsilon^x}(\xi) = \frac{d}{d\epsilon}\hat{\sigma_\epsilon^x}(\xi) = -2\epsilon^{-1} J_2\left(\epsilon\,|\xi|\right) e^{i(x,\xi)_{\mathbb{R}^4}}.$$

Therefore, both σ_ϵ^x and $d\sigma_\epsilon^x$ are in $H^{-2}\left(\mathbb{R}^4\right)$. In fact, for $\epsilon_1, \epsilon_2 > 0$,

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{\sigma_{\epsilon_2}^x}\right)\right] = \frac{1}{2\pi^2\epsilon_1\epsilon_2}\int_0^\infty \frac{\tau}{\left(1+\tau^2\right)^2} J_1\left(\epsilon_1\tau\right) J_1\left(\epsilon_2\tau\right) d\tau, \tag{7.1}$$

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{d\sigma_{\epsilon_2}^x}\right)\right] = \frac{-1}{2\pi^2\epsilon_1\epsilon_2}\int_0^\infty \frac{\tau^2}{\left(1+\tau^2\right)^2} J_1\left(\epsilon_1\tau\right) J_2\left(\epsilon_2\tau\right) d\tau, \tag{7.2}$$

and

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{d\sigma_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{d\sigma_{\epsilon_2}^x}\right)\right] = \frac{1}{2\pi^2\epsilon_1\epsilon_2}\int_0^\infty \frac{\tau^3}{\left(1+\tau^2\right)^2} J_2\left(\epsilon_1\tau\right) J_2\left(\epsilon_2\tau\right) d\tau. \tag{7.3}$$

Furthermore, for $x, y \in \mathbb{R}^4$, $x \neq y$, and $\epsilon_1, \epsilon_2 > 0$,

$$\begin{aligned} &\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{\sigma_{\epsilon_2}^y}\right)\right] \\ &\quad = \frac{1}{\pi^2\epsilon_1\epsilon_2\,|x-y|}\int_0^\infty \frac{1}{\left(1+\tau^2\right)^2} J_1\left(\epsilon_1\tau\right) J_1\left(\epsilon_2\tau\right) J_1\left(|x-y|\,\tau\right) d\tau, \end{aligned} \tag{7.4}$$

$$\begin{aligned} &\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{d\sigma_{\epsilon_2}^y}\right)\right] \\ &\quad = \frac{-1}{\pi^2\epsilon_1\epsilon_2\,|x-y|}\int_0^\infty \frac{\tau}{\left(1+\tau^2\right)^2} J_1\left(\epsilon_1\tau\right) J_2\left(\epsilon_2\tau\right) J_1\left(|x-y|\,\tau\right) d\tau, \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} &\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{d\sigma_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{d\sigma_{\epsilon_2}^y}\right)\right] \\ &\quad = \frac{1}{\pi^2\epsilon_1\epsilon_2\,|x-y|}\int_0^\infty \frac{\tau^2}{\left(1+\tau^2\right)^2} J_2\left(\epsilon_1\tau\right) J_2\left(\epsilon_2\tau\right) J_1\left(|x-y|\,\tau\right) d\tau. \end{aligned} \tag{7.6}$$

Proof. Everything follows from straightforward computations in spherical coordinates in $\mathbb{R}^4$ and applications of the following integral expression of the Bessel functions ([Wat], §3.3): for every $k \geq 1$ and $r > 0$,

$$J_k(r) = \frac{\left(\frac{r}{2}\right)^k}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^\pi e^{ir\cos(\theta)}\sin^{2k}\theta d\theta.$$

In addition, as we have indicated in Section 2, the asymptotic expansion ([Wat], §7.1) of J_k says that $J_k(r) = \mathcal{O}\left(r^{-1/2}\right)$ as $r \to \infty$, which is sufficient to guarantee the convergence of each integral involved in (7.1)-(7.6). □

It will be convenient to recognize that all the covariance functions involved in the previous lemma, i.e.,

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{\sigma_{\epsilon_2}^y}\right)\right], \mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{d\sigma_{\epsilon_2}^y}\right)\right] \text{ and } \mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{d\sigma_{\epsilon_1}^x}\right)\mathcal{I}\left(h_{d\sigma_{\epsilon_2}^y}\right)\right],$$

are continuous in all variables $x, y \in \mathbb{R}^4$ and $\epsilon_1, \epsilon_2 > 0$. In fact,

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{d\sigma^y_{\epsilon_2}}\right)\right] = \frac{d}{d\epsilon_2}\mathbb{E}\left[\mathcal{I}\left(h_{\sigma^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\sigma^y_{\epsilon_2}}\right)\right]$$

and

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{d\sigma^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{d\sigma^y_{\epsilon_2}}\right)\right] = \frac{d^2}{d\epsilon_1 d\epsilon_2}\mathbb{E}\left[\mathcal{I}\left(h_{\sigma^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\sigma^y_{\epsilon_2}}\right)\right].$$

These simply follow from the dominated convergence theorem and the fact that both $J_k(r)$ and $J_k(r)/r$ are bounded on $r \in (0, \infty)$ for every $k \geq 1$.

Proof of Lemma 1: The proof of all the formulas (2.3) to (2.5) is based on the following integral formulas of Bessel functions which can be found in [Wat], §13.53, pp 429-430: if $a \geq b > 0$ and $p > 0$, then

$$\int_0^\infty \frac{\tau}{\tau^2 + p^2} J_1(a\tau) J_1(b\tau)\, d\tau = K_1(ap) I_1(bp); \tag{7.7}$$

if $a > b + c$ and $p > 0$, then

$$\int_0^\infty \frac{1}{\tau^2 + p^2} J_1(a\tau) J_1(b\tau) J_1(c\tau)\, d\tau = p^{-1} K_1(ap) I_1(bp) I_1(cp). \tag{7.8}$$

We hereby provide an alternative proof of these two formulas, and complete the computations in Lemma 1. For a fixed $p > 0$, we define the function

$$B(a, b) \equiv \frac{1}{2\pi^2 ab}\int_0^\infty \frac{\tau}{\tau^2 + p^2} J_1(a\tau) J_1(b\tau)\, d\tau \text{ for } a, b > 0.$$

Given $a > 0$, we observe that if δ_x is the point mass at $x \in \mathbb{R}^4$, then the following integral is finite:

$$\begin{aligned}&\left(\frac{1}{2\pi}\right)^4 \int_{\mathbb{R}^4} \frac{1}{p^2 + |\xi|^2} \cdot e^{i(x,\xi)} \cdot \frac{2J_1(a|\xi|)}{a|\xi|} d\xi \\ =&\frac{1}{2\pi^2 a|x|}\int_0^\infty \frac{\tau J_1(a\tau) J_1(|x|\tau)}{p^2 + \tau^2} d\tau = B(a, |x|).\end{aligned}$$

But this integral is also "formally" equal to $\left(\left(p^2 - \Delta\right)^{-1}\sigma^0_a, \delta_x\right)_{L^2}$. In other words,

$$\left(\left(p^2 - \Delta\right)^{-1}\sigma^0_a\right)(x) = B(a, |x|)$$

is a point-wise defined, radially symmetric function in $x \in \mathbb{R}^4$. Therefore, in the sense of tempered distribution,

$$\left(p^2 - \Delta\right) B(a, |x|) = \sigma^0_a,$$

which, when written in spherical coordinates, implies

$$\left(p^2 - \partial_b^2 - \frac{3}{b}\partial_b\right) B(a, b) = 0 \text{ for all } 0 < b \neq a.$$

The above is a Bessel-type ordinary differential equation, all the solutions of which are in the form of

$$C_1(a)\frac{K_1(bp)}{b} + C_2(a)\frac{I_1(bp)}{b},$$

where C_1 and C_2 are two functions only depending on a. Without loss of generality, we can assume $b < a$. If one examines the behavior of $B(a, b)$ when b is close to zero, then one finds that $C_1(a) \equiv 0$ because $bB(a, b)$ converges to zero while $K_1(bp)$

blows up as $b \downarrow 0$. On the other hand, with $b > 0$ fixed, one can apply exactly the same arguments to see that $B(a,b)$ also satisfies

$$\left(p^2 - \partial_a^2 - \frac{3}{a}\partial_a\right) B(a,b) = 0 \text{ for all } a > b.$$

Hence, $C_2(a)$ must be in the form of

$$C_2(a) = C\frac{K_1(ap)}{a} + C'\frac{I_1(ap)}{a}$$

for some constant C and C'. This time, the boundedness of $aB(a,b)$ as $a \uparrow \infty$ implies $C' = 0$.

Thus, the only thing left is to determine the constant C. To this end, we observe

$$\begin{aligned} 2\pi^2 a^2 B(a,a) &= \int_0^\infty \frac{u}{u^2 + a^2p^2} J_1^2(u)\,du \\ &\longrightarrow \int_0^\infty \frac{J_1^2(u)}{u} du \text{ as } a \downarrow 0. \end{aligned}$$

However, one can easily verify that

$$\frac{d}{du}\left(-\frac{J_0^2(u) + J_1^2(u)}{2}\right) = \frac{J_1^2(u)}{u}.$$

So $\lim_{a\downarrow 0} 2\pi^2 a^2 B(a,a) = \frac{1}{2}$. Meanwhile, $\lim_{a\downarrow 0} I_1(ap)K_1(ap) = \frac{1}{2}$, which implies $C = \frac{1}{2\pi^2}$. Therefore,

$$B(a,b) = \frac{1}{2\pi^2 ab} I_1(bp)K_1(ap) \text{ for } a > b > 0.$$

For the formula (7.8), we define

$$C(a,b,c) = \frac{1}{\pi^2 abc}\int_0^\infty \frac{1}{\tau^2 + p^2} J_1(a\tau)J_1(b\tau)J_1(c\tau)\,d\tau \text{ for } a,b,c > 0.$$

Assume $a > b + c$. One can verify, by direct computations inside the integral signs and the dominated convergence theorem, that

$$p^2 C(a,b,c) - \frac{\partial^2}{\partial c^2}C(a,b,c) - \frac{3}{c}\frac{\partial}{\partial c}C(a,b,c) = 0 \text{ for } 0 < c < a - b,$$

$$\text{and } \lim_{c\downarrow 0} C(a,b,c) = B(a,b).$$

Similarly as above, one has

$$C(a,b,c) = \frac{2I_1(cp)}{cp}B(a,b) = \frac{1}{\pi^2 abcp}K_1(ap)I_1(bp)I_1(cp).$$

Thus, (7.7) and (7.8) are proved.

Given (7.7), notice that $\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\sigma^x_{\epsilon_2}}\right)\right]$ can be computed by applying the operator $\frac{-1}{2p}\frac{d}{dp}|_{p=1}$ to both sides of (7.7) with $a = \epsilon_1 \vee \epsilon_2$ and $b = \epsilon_1 \wedge \epsilon_2$. Then from there, based on the earlier observations, the complete expression for the

covariance matrix in the concentric case can be obtained by taking derivatives in ϵ_1 and ϵ_2 accordingly. The detailed computations are as follows. When $\epsilon_1 \geq \epsilon_2 > 0$,

$$\begin{aligned}\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\sigma^x_{\epsilon_2}}\right)\right] &= \left(-\frac{1}{2p}\frac{d}{dp}|_{p=1}\right)B\left(\epsilon_1,\epsilon_2\right)\\ &= \frac{-1}{4\pi^2}\left(K_1'\left(\epsilon_1\right)\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}+\frac{K_1\left(\epsilon_1\right)}{\epsilon_1}I_1'\left(\epsilon_2\right)\right)\\ &= \frac{-1}{4\pi^2}\begin{pmatrix} K_1'\left(\epsilon_1\right) & K_1\left(\epsilon_1\right)/\epsilon_1\end{pmatrix}\begin{pmatrix} I_1\left(\epsilon_2\right)/\epsilon_2\\ I_1'\left(\epsilon_2\right)\end{pmatrix}.\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbb{E}^{\mathcal{W}}\left[V^x_{\epsilon_1}\left(V^x_{\epsilon_2}\right)^\top\right]\\ =&\begin{pmatrix} 1 & \frac{\partial}{\partial\epsilon_2}\\ \frac{\partial}{\partial\epsilon_1} & \frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\end{pmatrix}\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\sigma^x_{\epsilon_2}}\right)\right]\\ =&\frac{-1}{4\pi^2}\begin{pmatrix} K_1'\left(\epsilon_1\right) & K_1\left(\epsilon_1\right)/\epsilon_1\\ K''\left(\epsilon_1\right) & \left(K_1\left(\epsilon_1\right)/\epsilon_1\right)'\end{pmatrix}\begin{pmatrix} I_1\left(\epsilon_2\right)/\epsilon_2 & \left(I_1\left(\epsilon_2\right)/\epsilon_2\right)'\\ I_1'\left(\epsilon_2\right) & I_1''\left(\epsilon_2\right)\end{pmatrix}.\end{aligned}$$

Besides, one realizes that

$$\left(\frac{K_1\left(\epsilon_1\right)}{\epsilon_1}\right)' = -\frac{K_2\left(\epsilon_1\right)}{\epsilon_1} \text{ and } \left(\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}\right)' = \frac{I_2\left(\epsilon_2\right)}{\epsilon_2},$$

and the formula (2.3) follows.

The non-concentric case is very similar. Given (7.8), we assign $a = \epsilon_1$ in case (2) and $a = |x-y|$ in case (3). By a similar procedure, i.e., applying $\frac{-1}{2p}\frac{d}{dp}|_{p=1}$ to (7.8) and taking derivatives in ϵ_1 and ϵ_2, we will be able to compute the non-concentric covariance matrix in either (2) or (3). To be specific, when $\epsilon_1 > |x-y| + \epsilon_2$, $\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\sigma^y_{\epsilon_2}}\right)\right]$ equals

$$\begin{aligned}&\left(-\frac{1}{2p}\frac{d}{dp}|_{p=1}\right)C\left(\epsilon_1,\epsilon_2,|x-y|\right)\\ =&\frac{-1}{2\pi^2}\left(K_1'\left(\epsilon_1\right)\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}\frac{I_1\left(|x-y|\right)}{|x-y|}+\frac{K_1\left(\epsilon_1\right)}{\epsilon_1}I_1'\left(\epsilon_2\right)\frac{I_1\left(|x-y|\right)}{|x-y|}\right)\\ &\qquad-\frac{1}{2\pi^2}\left(\frac{K_1\left(\epsilon_1\right)}{\epsilon_1}\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}I_1'\left(|x-y|\right)-\frac{K_1\left(\epsilon_1\right)}{\epsilon_1}\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}\frac{I_1\left(|x-y|\right)}{|x-y|}\right)\\ =&\frac{-1}{2\pi^2}\left(K_1'\left(\epsilon_1\right)\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}\frac{I_1\left(|x-y|\right)}{|x-y|}+\frac{K_1\left(\epsilon_1\right)}{\epsilon_1}I_1'\left(\epsilon_2\right)\frac{I_1\left(|x-y|\right)}{|x-y|}+\frac{K_1\left(\epsilon_1\right)}{\epsilon_1}\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}I_2\left(|x-y|\right)\right)\\ =&\frac{-1}{2\pi^2}\begin{pmatrix} K_1'\left(\epsilon_1\right) & \frac{K_1\left(\epsilon_1\right)}{\epsilon_1}\end{pmatrix}\begin{pmatrix}\frac{I_1\left(|x-y|\right)}{|x-y|} & 0\\ I_2\left(|x-y|\right) & \frac{I_1\left(|x-y|\right)}{|x-y|}\end{pmatrix}\begin{pmatrix} I_1\left(\epsilon_2\right)/\epsilon_2\\ I_1'\left(\epsilon_2\right)\end{pmatrix};\end{aligned}$$

when $|x-y| > \epsilon_1+\epsilon_2$, $\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\sigma^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\sigma^y_{\epsilon_2}}\right)\right]$ equals

$$
\begin{aligned}
&\left(-\frac{1}{2p}\frac{d}{dp}|_{p=1}\right) C\left(|x-y|,\epsilon_1,\epsilon_2\right)\\
=&\frac{-1}{2\pi^2}\left(K_1'\left(|x-y|\right)\frac{I_1\left(\epsilon_1\right)}{\epsilon_1}\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}-\frac{K_1\left(|x-y|\right)}{|x-y|}\frac{I_1\left(\epsilon_1\right)}{\epsilon_1}\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}\right)\\
&\qquad-\frac{1}{2\pi^2}\left(\frac{K_1\left(|x-y|\right)}{|x-y|}\frac{I_1\left(\epsilon_1\right)}{\epsilon_1}I_1'\left(\epsilon_2\right)+\frac{K_1\left(|x-y|\right)}{|x-y|}I_1'\left(\epsilon_1\right)\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}\right)\\
=&\frac{-1}{2\pi^2}\left(-K_2\left(|x-y|\right)\frac{I_1\left(\epsilon_1\right)}{\epsilon_1}\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}+\frac{K_1\left(|x-y|\right)}{|x-y|}\frac{I_1\left(\epsilon_1\right)}{\epsilon_1}I_1'\left(\epsilon_2\right)+\frac{K_1\left(|x-y|\right)}{|x-y|}I_1'\left(\epsilon_1\right)\frac{I_1\left(\epsilon_2\right)}{\epsilon_2}\right)\\
=&\frac{-1}{2\pi^2}\begin{pmatrix}\frac{I_1(\epsilon_1)}{\epsilon_1} & I_1'\left(\epsilon_1\right)\end{pmatrix}\begin{pmatrix}-K_2\left(|x-y|\right) & \frac{K_1(|x-y|)}{|x-y|}\\ \frac{K_1(|x-y|)}{|x-y|} & 0\end{pmatrix}\begin{pmatrix}I_1\left(\epsilon_2\right)/\epsilon_2\\ I_1'\left(\epsilon_2\right)\end{pmatrix}.
\end{aligned}
$$

The rest is straightforward. Thus we have finished the proof of Lemma 1. □

Before continuing, we point out that similar computations can be carried out in higher even dimensions $\mathbb{R}^{2n}$ with $n \geq 2$. In fact, the formulas (7.7) and (7.8) remain true if one replaces J_1 by J_{n-1} in the integrands, times a factor of τ^{2-n} to the integrand of (7.8), and replaces K_1 by K_{n-1}, I_1 by I_{n-1} respectively in the results. Therefore, one can use these modified results to compute the covariance function of the Gaussian family $\left\{\mathcal{I}\left(h_{\sigma^x_\epsilon}\right) : x \in \mathbb{R}^{2n}, \epsilon > 0\right\}$ (as defined in Section 6.1) by applying the operator $\left(-\frac{1}{2p}\frac{d}{dp}\right)^{n-1}|_{p=1}$ to the modified version of (7.7) and (7.8). The rest follows similarly as above.

Next, we want to provide details in the deriving the formulas for μ^ϵ_x (2.6) and $G(\epsilon)$ (2.7) as well as the results (2.8)-(2.10) in Theorem 2.6. It's an easy matter to check that for every $\epsilon > 0$,

$$\det \mathbf{B}\left(\epsilon\right) = \epsilon^{-1}\left(I_1^2\left(\epsilon\right) - I_0\left(\epsilon\right) I_2\left(\epsilon\right)\right) > 0,$$

where we applied the Bessel function identities ([Wat], §3.71) $I_1'\left(\epsilon\right) = \frac{-I_1(\epsilon)}{\epsilon} + I_0\left(\epsilon\right)$ and $I_1''\left(\epsilon\right) = \frac{-I_2(\epsilon)}{\epsilon} + I_1\left(\epsilon\right)$. Therefore,

$$\mathbf{B}^{-1}\left(\epsilon\right) = \frac{1}{I_1^2\left(\epsilon\right) - I_0\left(\epsilon\right) I_2\left(\epsilon\right)}\begin{pmatrix}\epsilon I_1\left(\epsilon\right) - I_2\left(\epsilon\right) & I_1\left(\epsilon\right) - \epsilon I_0\left(\epsilon\right)\\ -I_2\left(\epsilon\right) & I_1\left(\epsilon\right)\end{pmatrix}.$$

Recall that $U^x_\epsilon = \mathbf{B}^{-1}\left(\epsilon\right) V^x_\epsilon$, when computed explicitly,

$$U^x_\epsilon = \frac{1}{I_1^2\left(\epsilon\right) - I_0\left(\epsilon\right) I_2\left(\epsilon\right)}\begin{pmatrix}\left(\epsilon I_1\left(\epsilon\right) - I_2\left(\epsilon\right)\right)\mathcal{I}\left(h_{\sigma^x_\epsilon}\right) + \left(I_1\left(\epsilon\right) - \epsilon I_0\left(\epsilon\right)\right)\mathcal{I}\left(h_{d\sigma^x_\epsilon}\right)\\ -I_2\left(\epsilon\right)\mathcal{I}\left(h_{\sigma^x_\epsilon}\right) + I_1\left(\epsilon\right)\mathcal{I}\left(h_{d\sigma^x_\epsilon}\right)\end{pmatrix},$$

and if $\zeta = (1,1)^\top$, then $\mu^x_\epsilon = f_1\left(\epsilon\right)\sigma^x_\epsilon + f_2\left(\epsilon\right) d\sigma^x_\epsilon$ where

$$f_1\left(\epsilon\right) \equiv \frac{\epsilon I_1\left(\epsilon\right) - 2I_2\left(\epsilon\right)}{I_1^2\left(\epsilon\right) - I_0\left(\epsilon\right) I_2\left(\epsilon\right)} \text{ and } f_2\left(\epsilon\right) \equiv \frac{-\epsilon I_2\left(\epsilon\right)}{I_1^2\left(\epsilon\right) - I_0\left(\epsilon\right) I_2\left(\epsilon\right)}, \tag{7.9}$$

from which one sees that μ^x_ϵ has the "right" limit as $\epsilon \downarrow 0$. In addition, we can apply more Bessel function identities:

$$I_1'\left(\epsilon\right) = \frac{1}{\epsilon} I_1\left(\epsilon\right) + I_2\left(\epsilon\right),\ I_0\left(\epsilon\right) - I_2\left(\epsilon\right) = \frac{2}{\epsilon} I_1\left(\epsilon\right),$$

$$K_1'\left(\epsilon\right) = \frac{1}{\epsilon} K_1\left(\epsilon\right) - K_2\left(\epsilon\right) \text{ and } I_1\left(\epsilon\right) K_2\left(\epsilon\right) + I_2\left(\epsilon\right) K_1\left(\epsilon\right) = \frac{1}{\epsilon},$$

to write down $\mathbf{B}^{-1}(\epsilon)\,\mathbf{A}(\epsilon)$ explicitly as $\left(I_1^2(\epsilon)-I_0(\epsilon)\,I_2(\epsilon)\right)^{-1}$ times

$$\begin{pmatrix} -\left(1+\epsilon^{-2}\right) & I_1(\epsilon)\,K_1(\epsilon)+I_2(\epsilon)\,K_0(\epsilon)+\epsilon^{-2} \\ I_1(\epsilon)\,K_1(\epsilon)+I_2(\epsilon)\,K_0(\epsilon)+\epsilon^{-2} & -\epsilon^{-2} \end{pmatrix}.$$

Therefore we have:
(1), given $x\in\mathbb{R}^4$ and $\epsilon_1\geq\epsilon_2>0$,

$$\begin{aligned}\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\mu^x_{\epsilon_2}}\right)\right]=G(\epsilon)&\equiv\left(-\frac{1}{4\pi^2}\right)\zeta^\top\mathbf{B}^{-1}(\epsilon)\,\mathbf{A}(\epsilon)\,\zeta\ (\text{with }\zeta^\top=(1,1))\\&=\left(-\frac{1}{4\pi^2}\right)\frac{2I_1(\epsilon)\,K_1(\epsilon)+2I_2(\epsilon)\,K_0(\epsilon)-1}{I_1^2(\epsilon)-I_0(\epsilon)\,I_2(\epsilon)};\end{aligned}$$

(2), given $x,y\in\mathbb{R}^4$, $x\neq y$, and $\epsilon_1,\epsilon_2>0$ with $\epsilon_1>|x-y|+\epsilon_2$, $\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\mu^x_{\epsilon_2}}\right)\right]$ equals $\frac{-1}{2\pi^2}\left(I_1^2-I_0I_2\right)^{-1}(\epsilon_1)$ times

$$\begin{aligned}&\zeta^\top\mathbf{B}^{-1}(\epsilon_1)\,\mathbf{A}(\epsilon_1)\,\mathbf{C}(|x-y|)\,\zeta.\\=&\begin{pmatrix}(I_1K_1+I_2K_0)(\epsilon_1)-1 & (I_1K_1+I_2K_0)(\epsilon_1)\end{pmatrix}\begin{pmatrix}\frac{I_1(|x-y|)}{|x-y|}\\ I_2(|x-y|)+\frac{I_1(|x-y|)}{|x-y|}\end{pmatrix}\\=&\left(\frac{2I_1(|x-y|)}{|x-y|}+I_2(|x-y|)\right)\left((I_1K_1+I_2K_0)(\epsilon_1)-\frac{1}{2}\right)+\frac{1}{2}I_2(|x-y|)\\=&I_0(|x-y|)\left((I_1K_1+I_2K_0)(\epsilon_1)-\frac{1}{2}\right)+\frac{1}{2}I_2(|x-y|);\end{aligned}$$

(3), given $x,y\in\mathbb{R}^4$, $x\neq y$, and $\epsilon_1,\epsilon_2>0$ with $|x-y|>\epsilon_1+\epsilon_2$,

$$\begin{aligned}\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\mu^x_{\epsilon_2}}\right)\right]&=\left(-\frac{1}{2\pi^2}\right)\zeta^\top\mathbf{D}(|x-y|)\,\zeta.\\&=\left(-\frac{1}{2\pi^2}\right)\left(\frac{2K_1(|x-y|)}{|x-y|}-K_2(|x-y|)\right)\\&=\frac{1}{2\pi^2}K_0(|x-y|).\end{aligned}$$

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